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Phase portraits of uniform isochronous quartic centers

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In this paper we classify the global phase portraits in the Poincaré disc of all quartic polynomial differential systems with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous.

1. Introduction and statement of the main results

Christian Huygens is credited with being one of the first scholars to study isochronous systems in the XVII century, even before the development of the differential calculus. Huygens investigated the cycloidal pendulum, which has isochronous oscillations in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochrone. For more details see [1].

Isochronicity appears in a wide variety of Physics phenomena and it is also closely related to the uniqueness and existence of solutions for some boundary value, perturbation, or bifurcation problems. Moreover it is important in stability theory, since a periodic solution in the region surrounding the center type singular point is Liapunov stable if and only if the neighboring periodic solutions have the same period. For more details on these topics see [2]. In the last decades the study of isochronous systems has been increased due to the proliferation of powerful methods of computerized research, and special attention has been dedicated to polynomial differential systems, see [3–6] and the bibliography therein.

In this paper we classify the global phase portraits of all quartic polynomial differential systems with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous.

Let $p \in \mathbb{R}^2$ be a center of a differential polynomial system in $\mathbb{R}^2$, without loss of generality we can assume that p is the origin of coordinates. We say that p is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that p is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, takes the form $\dot r = G(\theta, r), \dot\theta = k,\ k \in \mathbb{R}\setminus\{0\}$, for more details see Conti [5].

Proposition 1. *Assume that a planar differential polynomial system $\dot x = P(x, y),\ \dot y = Q(x, y)$ of degree n has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written into the form*

$$\dot x = -y + x f(x, y), \qquad \dot y = x + y f(x, y), \tag{1}$$

where $f(x, y)$ is a polynomial in x and y of degree $n - 1$, and $f(0, 0) = 0$.

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