Ergod. Th. & Dynam. Sys. (2000), **20**, 775–799
Printed in the United Kingdom

Hamiltonian stability in the plane

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(*Received* 4 *April* 1998 *and accepted in revised form* 30 *July* 1998)

Abstract. We characterize the Hamiltonian flows in the plane which are structurally stable (in a global sense) among Hamiltonian flows. This notion is closely related to, but distinct from, the topological stability of the generating function as a map from the plane to the line.

1. *Introduction and statement of results*

In this paper we give a dynamic characterization of the Hamiltonian flows whose phase space is the plane $\mathbb{R}^2$ which possess a certain global kind of structural stability.

The essentially unique topology for the space of continuous functions on a compact space leads to a natural notion of structural stability for dynamical systems on a closed manifold. However, the continuous functions on a non-compact space have a number of natural topologies. As a result, several distinct versions of structural stability can be formulated for dynamical systems on an open manifold, such as the plane. The comparative discussion of these notions in [**KKN**] gives a rationale for the version we shall adopt here.

For r a non-negative integer, $\mathcal{F}^r$ denotes the set of $\mathcal{C}^r$ functions $f : \mathbb{R}^2 \to \mathbb{R}$. Given $f \in \mathcal{F}^r$ and $x \in \mathbb{R}^2$, let $\|f(x)\|_{\mathcal{C}^r}$ be the maximum among the absolute values of $f(x)$ and its partial derivatives up to and including order r, all evaluated at x. A basis for the neighborhoods of $f \in \mathcal{F}^r$ in the *strong $\mathcal{C}^r$ topology* of Whitney is given by the sets

$$\mathcal{N}_\varepsilon(f) = \{g \in \mathcal{F}^r \mid \|g(x) - f(x)\|_{\mathcal{C}^r} < \varepsilon(x)\ \forall x \in \mathbb{R}^2\}$$

where $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^+ = (0, \infty)$ ranges over positive functions on $\mathbb{R}^2$. The space $\mathcal{X}^r$ of $\mathcal{C}^r$ vectorfields is topologized by applying these estimates componentwise.

A *dynamical equivalence* between two flows Φ and Ψ on $\mathbb{R}^2$ is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking directed Φ-trajectories to directed Ψ-trajectories. Given $K \subset U \subset \mathbb{R}^2$

§ Work done while on a postdoctoral year at Boston University, supported by the Ministerio de Educación y Cultura of Spain.

with K compact and U open, we say $h \in \mathcal{N}_{\rm co}(K,U)$ (a *compact-open neighborhood of the identity*) if $h(K) \subset U$. Note that, given K and U, a subset of $\mathcal{N}_{\rm co}(K,U)$ is specified by an estimate of the form

$$|h(x) - x| < \delta, \quad \forall x \in K$$

where δ is less than the distance from K to the exterior of U.

The notion of structural stability in [**KKN**] is given by the following.

Definition 1. A $\mathcal{C}^r$ flow Φ with velocity vectorfield X is *globally $\mathcal{C}^r$ structurally stable* if given a compact-open neighborhood $\mathcal{N}_{\rm co}(K,U)$ of the identity, there exists a basic neighborhood $\mathcal{N}_\varepsilon(X)$ of X such that every flow Ψ with velocity vectorfield in $\mathcal{N}_\varepsilon(X)$ is dynamically equivalent to Φ, with $h \in \mathcal{N}_{\rm co}(K,U)$.

Globally $\mathcal{C}^r$ structurally stable flows in the plane ($r \geq 1$) were characterized in [**KKN**], giving a kind of extension of Peixoto's classic structural stability theorem for flows on closed surfaces. A function $f : \mathbb{R}^2 \to \mathbb{R}$ generates a *Hamiltonian vectorfield* X_f (respectively, *Hamiltonian flow* Φ_f) via the Hamiltonian system of o.d.e.'s

$$\frac{dx}{dt} = \frac{\partial f}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial f}{\partial x}.$$

Observe that the generating function f is determined up to an additive constant by X_f, and the passage from f to X_f drops one degree of differentiability.

In [**JL1, JL2**], the Hamiltonian flows which are globally $\mathcal{C}^r$ structurally stable were characterized. These results address a strong kind of stability, which requires the Hamiltonian flow to be equivalent to all nearby flows, Hamiltonian or not. A natural weakening of this notion considers only perturbations within the subspace $\mathcal{H}^r \subset \mathcal{X}^r$ of Hamiltonian vectorfields. $\mathcal{H}^r$ inherits the strong $\mathcal{C}^r$ topology from $\mathcal{X}^r$, but this can also be formulated in terms of the strong $\mathcal{C}^{r+1}$ topology on the generating functions.

Definition 2. A function $f \in \mathcal{F}^{r+1}$ (respectively, Hamiltonian flow $\Phi_f \in \mathcal{H}^r$ with velocity vectorfield X_f) is *Hamiltonian $\mathcal{C}^r$ stable* if given a compact-open neighborhood $\mathcal{N}_{\rm co}(K,U)$ of the identity, there exists a strong $\mathcal{C}^r$ neighborhood $\mathcal{N}_\varepsilon(X_f) \cap \mathcal{H}^r$ of X_f in $\mathcal{H}^r$ (equivalently, a strong $\mathcal{C}^{r+1}$ neighborhood $\mathcal{N}_{\varepsilon'}(f)$ in $\mathcal{F}^{r+1}$) such that every Hamiltonian flow Φ_g with $X_g \in \mathcal{N}_\varepsilon(X_f)$ (equivalently $g \in \mathcal{N}_{\varepsilon'}(f)$) is dynamically equivalent to Φ_f, with $h \in \mathcal{N}_{\rm co}(K,U)$.

In [**JL1**], related notions of stability for Hamiltonian flows generated by polynomial functions were considered, using the strong $\mathcal{C}^r$ topology and the coefficient topology, respectively, to define neighborhoods of f within the subspace of polynomial functions.

In this paper, we characterize the Hamiltonian stable flows (or functions) in the plane. We shall see that Hamiltonian $\mathcal{C}^1$ stability is equivalent, for a $\mathcal{C}^r$ flow (respectively, $\mathcal{C}^{r+1}$ function), to Hamiltonian $\mathcal{C}^r$ stability for all $r \geq 1$ (the latter is *a priori* weaker), and so our results are formulated without explicit reference to r, which is assumed positive. Examples of Hamiltonian stable flows which are not globally structurally stable abound (see below).

The generating function f of a Hamiltonian flow Φ_f is automatically constant along trajectories, so that the dynamic structure of the flow is closely related to the structure of the *level sets*

$$\mathcal{L}_c = \mathcal{L}_c(f) = \{p \in \mathbb{R}^2 \mid f(p) = c\}$$

of the function. A level set corresponding to a regular value is a disjoint union of simple curves, and when all critical points are non-degenerate (see §2), a critical level is also a union of curves, which may cross themselves at critical points. We will refer to any component of a level set of f as a *level curve* for f.

By the invariance of f, every trajectory of Φ_f is contained in a level curve of f; in fact, such a trajectory is either equal to the level curve containing it, or else its boundary points in this level curve are critical points of f. From this we can formulate the following correspondence between the dynamics of Φ_f and the level curve structure of f.

Remark 1. If the critical points of $f \in \mathcal{F}$ form a totally disconnected set, and $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, then h takes the level curves of f to the level curves of some function $g \in \mathcal{F}$ if and only if h is a dynamical equivalence between Φ_f and one of the two flows Φ_g, Φ_{-g} (which differ by time reversal).

This means that Hamiltonian $\mathcal{C}^r$ stability of $f \in \mathcal{F}^{r+1}$ is closely related to the topological stability of f as a $\mathcal{C}^{r+1}$ mapping, for which there exists an elegant theory [**dPW**]. However, there is a subtle but substantive difference between the notions of equivalence in these two theories. Two mappings $f, g : \mathbb{R}^2 \to \mathbb{R}$ are topologically equivalent in the sense of [**dPW**] if there exist homeomorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$g \circ h = \phi \circ f.$$

Again assuming some non-degeneracy (as in Remark 1) the existence of ϕ is guaranteed provided we have $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking level *sets* of f to level *sets* of g, and this property is required of h. However, in Remark 1, distinct level *curves* of f inside the same level *set* are allowed to map to level curves of g which are contained in distinct level sets. We shall explore this distinction further in §5, and in particular will in Proposition 5 produce a non-empty open set of functions which are Hamiltonian stable (in our sense) but fail to be topologically stable as maps (in the sense of [**dPW**]).

Nonetheless, the close connection between these notions of stability allows us to adapt to our purposes several ideas from [**dPW**]. We are indebted to James Montaldi for making us aware of [**dPW**], and to Andrew du Plessis for directing our attention to [**dPW**, Theorem 3.6.1], which greatly influenced our proof of the sufficiency of our conditions.

To formulate our result, we recall some dynamical notions. Fix $f \in \mathcal{F}^2$, and the related $\mathcal{C}^1$ Hamiltonian vectorfield (respectively, flow) X_f (respectively, Φ_f). An equilibrium point p of Φ_f is a zero of X_f, or equivalently a critical point of f. This point is called hyperbolic if the eigenvalues of the linearization matrix at p of X_f

$$\begin{pmatrix} \dfrac{\partial X_1}{\partial x}(p) & \dfrac{\partial X_1}{\partial y}(p) \\ \dfrac{\partial X_2}{\partial x}(p) & \dfrac{\partial X_2}{\partial y}(p) \end{pmatrix}$$

have non-zero real parts. When X is Hamiltonian, the characteristic polynomial of this matrix is $\lambda^2 - \mathcal{D}f$, where $\mathcal{D}f$ is the discriminant of f, so the cases of sink and source (where both eigenvalues are on the same side of the imaginary axis) cannot occur. The only hyperbolic possibility is a *hyperbolic saddle*, where the eigenvalues are real and satisfy

$\lambda_s < 0 < \lambda_u$. This corresponds in the Hamiltonian case to a non-degenerate critical point of f which is not a local extremum. The well-known local phase portrait near a saddle involves two pairs of 'separatrix trajectories': one pair of *stable separatrices*, tending toward the equilibrium (and asymptotically tangent to the eigenspace of λ_s) as $t \to \infty$, the other *unstable separatrices*, tending to equilibrium along λ_u as $t \to -\infty$.

For a dense open subset of $\mathcal{X}$, every equilibrium is hyperbolic. However, another kind of equilibrium, corresponding to a local extremum of f, is not removable within $\mathcal{H}$. Generically in $\mathcal{H}$, this is a *non-degenerate center*, where the eigenvalues are a pair of complex-conjugate, non-zero, pure, imaginary numbers, and the local phase portrait consists of nested periodic orbits surrounding the equilibrium point.

The relevant behavior at infinity is formulated as in [**KKN**] (following Nemytskii–Stepanov [**NS**]). A point $p \in \mathbb{R}^2$ *escapes to infinity* in positive (respectively, negative) time if for every compact set $K \subset \mathbb{R}^2$ there exists a time $T = T(K, p) \in \mathbb{R}$ such that $\Phi^t(p)$ is not in K for $t > T$ (respectively, $t < T$). A point $q \in \mathbb{R}^2$ belongs to the positive *prolongational limit set* of p if there exist $p_i \to p$ in $\mathbb{R}^2$ and $t_i \to +\infty$ in $\mathbb{R}$ such that $\Phi^{t_i}(p_i) \to q$; we write $q \in \mathcal{J}_+(p)$, or equivalently $p \in \mathcal{J}_-(q)$, in this case. It is easy to see that this is invariant under the flow: for each $t \in \mathbb{R}$,

$$\Phi^t(\mathcal{J}_\pm(p)) = \mathcal{J}_\pm(p) = \mathcal{J}_\pm(\Phi^t(p)).$$

The local phase portrait near a hyperbolic saddle yields the following, which we formalize for later reference.

Remark 2. Suppose s is a hyperbolic saddle of the flow Φ.

(1) If $p_i \to p$ with p_i on distinct orbits, and p belongs to a stable (respectively, unstable) separatrix of s, then there exist $q_i \to q$ with q_i on the orbit of p_i and q belonging to an unstable (respectively, stable) separatrix of s.

(2) If $q \in \mathcal{J}_+(p)$, where q belongs to a stable or unstable separatrix of s, then $\mathcal{J}_+(p)$ includes at least one stable and one unstable separatrix of s.

(3) If p belongs to a stable separatrix of s and q belongs to an unstable separatrix of s, then $q \in \mathcal{J}_+(p)$.

We say that a *saddle at infinity* occurs whenever $q \in \mathcal{J}_+(p)$ and p (respectively, q) escapes to infinity in forward (respectively, backward) time; the orbit of p (respectively, q) is then the *stable separatrix* (respectively, *unstable separatrix*) of this (infinite) saddle. We refer to hyperbolic saddle points as 'finite saddles'. A *saddle connection* between two saddles (finite or infinite, and not necessarily distinct) is a trajectory which is simultaneously a stable separatrix of one saddle and an unstable separatrix of the other. Special cases are: a *homoclinic contour*, a saddle connection between a finite saddle and itself, and a *homoclinic contour at infinity*—an orbit escaping to infinity in both time directions which belongs to its own prolongational limit set (in other words, a non-wandering orbit with empty α- and ω-limit sets).

Examples of vectorfields exhibiting these phenomena are given in [**KKN**]. Here we note some Hamiltonian examples.

Consider first the function (Figure 1)

$$f(x, y) = y^2 - \frac{x^2}{x^4 + 1},$$

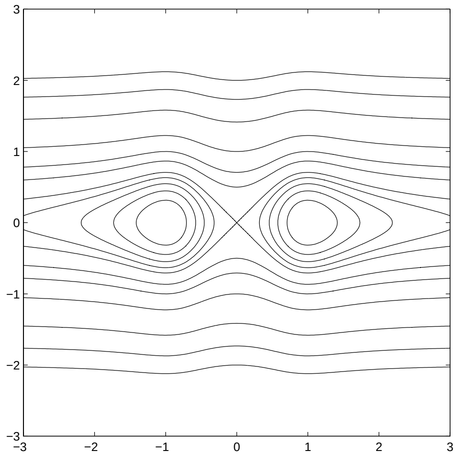

FIGURE 1. Finite-to-infinite saddle connection.

which has an absolute minimum value $-1/2$, achieved at the pair of points $(\pm 1, 0)$, and a saddle critical point at the origin, with value zero. For $1/2 < c < 0$, the level set $\mathcal{L}_c(f)$ consists of a pair of ovals, one around each of the minimum points. The level set $\mathcal{L}_0(f)$ consists of the four separatrices of the origin, two asymptotic to each of the positive and negative x-axis. Each such pair forms a saddle at infinity (the ovals contain the points p_i, q_i in the definition). For $c > 0$, the level set $\mathcal{L}_c(f)$ consists of the two nearly horizontal curves $y = \pm[c + x^2/(x^4 + 1)]^{1/2}$. Thus, we have a saddle connection between the finite saddle at the origin and the two saddles at infinity.

Second, consider the polynomial function (Figure 2)

$$f(x, y) = x + x^2 y.$$

This has no critical points, but the y-axis forms a saddle at infinity with each branch of the hyperbola $xy = -1$, and hence we have a saddle connection between two saddles at infinity.

Third, consider the function (Figure 3)

$$f(x, y) = \frac{y}{(x^2 + 1)(y^2 + 1)}.$$

This has an absolute maximum at $(0, 1)$ (with value $1/2$) and an absolute minimum at $(0, -1)$ (with value $-1/2$). The level set $\mathcal{L}_0(f)$ is the x-axis, and for $0 < |c| < 1/2$ the level set $\mathcal{L}_c(f)$ is an oval surrounding one of the two extrema noted above. Clearly, each point p on the x-axis satisfies $p \in \mathcal{J}_+(p)$, and so the x-axis is a homoclinic contour at infinity.

Fourth, consider the function

$$f(x, y) = y^2 + 4x^3 - 3x^2.$$

This has a relative minimum at $(1/2, 0)$ (with value $-1/4$) and a saddle point at the origin with value zero. The level set $\mathcal{L}_0(f)$ is a curve crossing itself at the origin, and forming

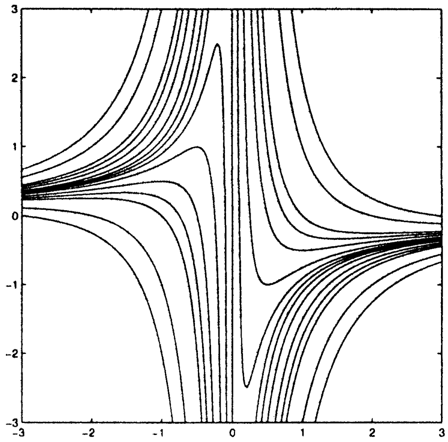

FIGURE 2. Connection between saddles at infinity.

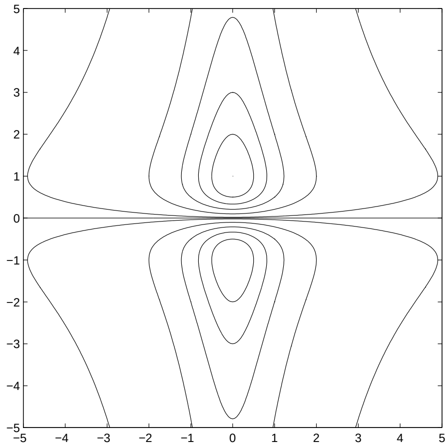

FIGURE 3. Homoclinic contour at infinity.

a loop around $(-1/2, 0)$, and with two branches escaping to infinity (with $y \to \pm\infty$, respectively) in the left half-plane. The loop is a homoclinic contour for the saddle at the origin. There are no saddles at infinity (Figure 4).

Finally, we sketch a way to use this last example to construct accumulating separatrices as in [**KKN**]; note that this phenomenon cannot occur for polynomial functions. Let γ_0 be the branch of $\mathcal{L}_0(f)$ in the lower (left) half-plane

$$\gamma_0 : \quad y = x\sqrt{4x-3}, \quad x < 0$$

and γ_1 the part of $\mathcal{L}_1(f)$ in the lower half-plane:

$$\gamma_1 : \quad y = -\sqrt{1+3x^2-4x^3}, \quad x < 1.$$

The region R of the lower half-plane between these curves is diffeomorphic to the strip

$$S_0 = [0, 1] \times (-\infty, \infty)$$

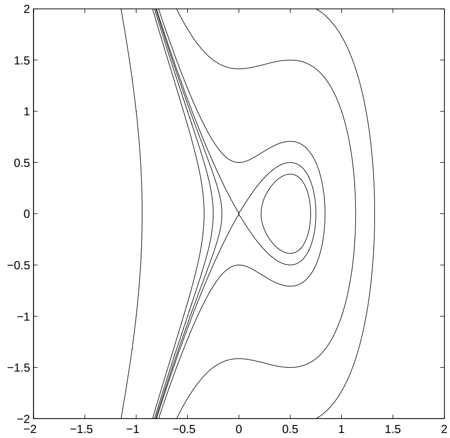

FIGURE 4. Homoclinic contour.

with γ_α ($\alpha = 0, 1$) mapping to $\{\alpha\} \times (-\infty, \infty)$. This transfers f to a function F_0 on S_0. By adjusting the diffeomorphism, we can assume that:

(1) F_0 is $\mathcal{C}^\infty$, and its value and all partial derivatives at each point of the boundary agree with those of the function $(x, y) \mapsto x$;

(2) for $y \leq 0$, $F_0(x, y) = x$.

Note that the diffeomorphism maps the two level curves in $\mathcal{L}_0(f) \cap R$ to a saddle at infinity for Φ_{F_0} in S_0, with one separatrix along the y-axis and the other contained in the upper half-plane; this gives us a template similar to [**KKN**, Figure 2.8(a)]. Now, we can create a similar template function F_n, $n = 1, \ldots,$ defined on $S_n = [0, 1/2^n] \times (-\infty, \infty)$, by scaling and translation:

$$F_n(x, y) = 2^{-n} F_0(2^n x, y - n).$$

Note that F_n also satisfies the first property above, as well as the second, but for $y \leq n$, and the left edge forms a saddle at infinity with a curve lying above the line $y = n$. Now, we can define a new function F on $\mathbb{R}^2$ by

$$F(x, y) = \begin{cases} x, & x \leq 0 \text{ and } x \geq 1 \\ \dfrac{1}{2^n} + F_n\left(x - \dfrac{1}{2^n}, y\right), & \dfrac{1}{2^n} \leq x \leq \dfrac{1}{2^{n-1}}. \end{cases}$$

Then it is easy to check from the conditions on F_n above that $F(x, y)$ is a well defined $\mathcal{C}^\infty$ function on $\mathbb{R}^2$. Each line $x = 1/2^n$ forms a saddle at infinity with a curve in the region $1/2^n < x < 1/2^{n-1}$, $y \geq n$; in particular, the y-axis is an accumulation of separatrices of saddles at infinity for Φ_F. By Theorems 1 and 3 below, this phenomenon does not prevent structural or Hamiltonian stability. We can, however, use the same procedure to create more complicated examples of accumulation of separatrices by replacing any 'parallelizable' region of a Hamiltonian flow with a copy of F on $[0, 1] \times (-\infty, \infty)$; for example, we can create examples in which separatrices (of saddles at infinity) accumulate at a separatrix of a finite saddle or one in which both stable and unstable separatrices (of saddles at infinity) accumulate on a non-separatrix orbit. Theorem 3 below tells us that the former kind of example is not even Hamiltonian stable, while the latter is Hamiltonian stable, but (by Theorem 1) not structurally stable.

The globally structurally stable flows in $\mathbb{R}^2$ are characterized by the following.

THEOREM 1. [**KKN**] *A flow in the plane is globally $\mathcal{C}^r$ structurally stable (for $r \geq 1$) if and only if all the following hold:*

(1) *every equilibrium is hyperbolic,*
(2) *every periodic orbit is a non-degenerate limit cycle,*
(3) *every unbounded semi-orbit escapes to infinity, and*
(4) *the closure of the set of all stable separatrices (of finite or infinite saddles) intersects the closure of the set of all unstable separatrices precisely in the set of (finite) hyperbolic saddle points.*

These conditions can be simplified when the flow is Hamiltonian.

THEOREM 2. [**JL2**] *A Hamiltonian flow Φ_f in the plane is globally $\mathcal{C}^r$ structurally stable ($r \geq 1$) if and only if*

(1) *every equilibrium is a hyperbolic saddle, and*
(2) *the closure of the set of all stable separatrices (of finite or infinite saddles) intersects the closure of the set of all unstable separatrices precisely in the set of equilibria.*

To see this, observe the following.

- A Hamiltonian flow has no sinks or sources, so a hyperbolic equilibrium is automatically a saddle. Note that the first condition in Theorem 2 thus rules out relative extrema for f.
- Since a periodic orbit of Φ_f must enclose a relative extremum of f, the first condition in Theorem 2 already rules out periodic orbits.
- While it is possible to construct a Hamiltonian flow in $\mathbb{R}^2$ with an unbounded semi-orbit that does not escape to infinity, such an example requires a curve of critical points, and this can be avoided generically.

Our result on Hamiltonian stability can be formulated in terms of either the dynamics of Φ_f or the level curve structure of f. The following captures, in terms of the level curves of f, the notion of a 'saddle at infinity' for Φ_f.

Definition 3. $p \in \mathbb{R}^2$ is *virtually critical* for $f \in \mathcal{F}$ if there exists a sequence of embedded intervals C_i with endpoints p_i, y_i, on each of which f has a constant value distinct from $f(p)$, with $p_i \to p$, y_i convergent, but some sequence $r_i \in C_i$ has no accumulation points in $\mathbb{R}^2$.

If p and q belong, respectively, to the stable and unstable separatrix of some saddle at infinity, then the sequence of segments of Φ_f-trajectories C_i gives the above condition for p (and for q). Conversely, we shall prove in §2, Lemma 3 that for a Morse function, the above situation implies the existence of a saddle at infinity whose separatrices intersect the level curves through p and q. Note that if p is virtually critical (and all critical points are non-degenerate), then so is every point on the level curve through p.

Definition 4. By a *critical curve* for $f \in \mathcal{F}$ we mean a level curve containing a critical point. A critical curve C is *proper* if there is a neighborhood V of C (which we call an *isolating neighborhood* for C) such that:

(1) V contains a unique critical point (which must belong to C);

(2) V contains no virtually critical points;

(3) C is the only critical curve intersecting V.

Our main result can then be formulated as follows (the equivalence of the dynamic and functional formulations of the second condition will follow from Lemma 3).

THEOREM 3. *For $r \geq 1$, suppose $f \in \mathcal{F}^{r+1}$. Then f (respectively, the Hamiltonian flow Φ_f generated by f) is Hamiltonian $\mathcal{C}^r$ stable if and only if:*

(1) *every critical point of f is non-degenerate (equivalently, every equilibrium of Φ_f is either a hyperbolic saddle or a non-degenerate center);*

(2) *every critical curve is proper (equivalently, a separatrix of a finite saddle is isolated from the separatrices of all other finite or infinite saddles for Φ_f).*

Observe the following phenomena which are prohibited in Theorem 2, but allowed in Theorem 3:

- (non-degenerate) relative extrema of f;
- homoclinic contours;
- periodic orbits (hence compact level curves);
- regular points, not lying on any separatrix of a finite saddle, which are limits of a pair of sequences, one sequence contained in stable separatrices of (finite or infinite) saddles, the other in unstable separatrices.

We will prove this theorem in the next three sections, as follows. In §2, we sketch the necessary modifications to adapt to the plane the standard proofs of density of Morse functions on compact manifolds, and as a consequence prove the necessity of the first condition, and those parts of the second dealing only with critical curves, for Hamiltonian stability. In §3 we complete the proof of necessity of the second condition via a study of saddles at infinity for Hamiltonian flows. In §4 we prove sufficiency of these conditions for Hamiltonian stability.

Finally, in §5 we clarify the distinction between Hamiltonian stability and topological stability for a planar function by constructing a non-empty open set of Hamiltonian stable functions which fail to be topologically stable.

2. *Morse functions*

In this section, we use well known generic properties of functions to prove the necessity of the first condition, and part of the second, in Theorem 3.

The generic structure of a real-valued function on a compact manifold is well known. Denote by $\mathrm{crit}(f)$ the set of critical points of f. At any $p \in \mathrm{crit}(f)$, the hessian $H_p f$ is a symmetric bilinear form given in local coordinates by the matrix of second-order partial derivatives of f at p. The critical point is *non-degenerate* if this matrix is non-singular: p is a local extremum of f if $H_p f$ is positive (or negative) definite, and otherwise is a non-degenerate saddle. A function f is a *Morse function* if

(1) every critical point of f is non-degenerate, and
(2) distinct critical points belong to distinct level sets of f.

A standard application of transversality arguments (e.g. [**H**]) shows that Morse functions form a dense open set in the space of $\mathcal{C}^r$ functions on a compact manifold (for $r \geq 2$). These arguments are a combination of local estimates and applications of the Baire category theorem, and can be adapted to show density in the case of open manifolds. In the plane, the delicate constructions found in most transversality theorems are not needed, so we sketch the argument directly in this case.

LEMMA 1. *The functions $f : \mathbb{R}^2 \to \mathbb{R}$, for which every critical point is non-degenerate, form a dense open subset of $\mathcal{F}^r$ for $r \geq 2$.*

Proof. (Sketch) Note first that if p is a critical point of $f \in \mathcal{F}^r$, we have, using subscript notation for derivatives ($f_1 = \partial f/\partial x$, $f_2 = \partial f/\partial y$) that $f_1(p) = 0 = f_2(p)$, and the critical point is non-degenerate if the discriminant

$$\mathcal{D}f(p) = f_{1,1}(p)f_{2,2}(p) - f_{1,2}(p)^2$$

is non-zero at p. Note that this expression makes sense at any point in the plane, not just at critical points.

Openness: A non-degenerate critical point is isolated in $\mathrm{crit}(f)$, so we can cover $\mathrm{crit}(f)$ with a finite or countable family of open discs B_i, $i \in \mathbb{N}$, such that the unique critical point in B_i is its center, p_i. By shrinking these discs, we can assume that $\mathcal{D}f(p)$ is bounded away from zero on each B_i. Now, cover the complement of $\bigcup B_i$ with open sets U_j, $j \in \mathbb{N}$, such that f_1 and f_2 are bounded away from zero on each U_j, and so that the B_i's and U_j's form a locally finite cover of $\mathbb{R}^2$. For each B_i, we can find $\varepsilon_i > 0$ such that any function g whose values and first and second derivatives differ from those of f, pointwise on B_i, by less than ε_i, has a unique critical point q_i in B_i, and $\mathcal{D}g \neq 0$ on B_i. Note for later reference that by reducing ε_i, we can also ensure, given $\delta_i > 0$, that $|f(p_i) - g(q_i)| < \delta_i$. For each U_j, we can find $\varepsilon_j > 0$ such that a function with first derviatives differing from those of f by less than ε_j (pointwise on U_j) has no critical points in U_j. Now, the conditions $0 < \varepsilon(p) < \varepsilon_i$ on B_i and $0 < \varepsilon(p) < \varepsilon_j$ on U_j are locally finite, so we easily find $\varepsilon : \mathbb{R}^2 \to (0, \infty)$ continuous satisfying all these conditions. Clearly, each $g \in \mathcal{N}_\varepsilon(f)$ has only the critical points q_i, and they are non-degenerate.

Density: If the origin in $\mathbb{R}^2$ is a degenerate critical point of f, then for any $(\varepsilon, \delta) \in \mathbb{R}\times\mathbb{R}$ the function

$$g(x, y) = f(x, y) + \frac{\varepsilon}{2}x^2 + \frac{\delta}{2}y^2$$

has a critical point at the origin, with the same critical value as f, but with

$$\begin{aligned} g_{1,1}(0, 0) &= f_{1,1}(0, 0) + \varepsilon \\ g_{2,2}(0, 0) &= f_{2,2}(0, 0) + \delta \\ g_{1,2}(0, 0) &= f_{1,2}(0, 0). \end{aligned}$$

Using a bump function, we can create a function which agrees with g in a neighborhood of the origin and with f outside a slightly larger neighborhood, and given these neighborhoods, we can adjust ε and δ so that this new function is inside any specified basic neighborhood of f in $\mathcal{F}^r$. Clearly, a similar operation can be carried out at any critical point of f. A Baire category argument (which we omit) then completes the proof of density. $\square$

Since non-degenerate critical points are isolated and $\mathrm{crit}(f)$ is closed, a function on a compact manifold with no degenerate critical points has $\mathrm{crit}(f)$ finite. If two critical points of such a function have the same critical value, we can add a small bump function near one of them to separate these values. The openness of Morse functions on a compact space follows immediately. In the plane, we can only conclude that a function without degenerate points has $\mathrm{crit}(f)$ discrete, and the set of critical values $f(\mathrm{crit}(f))$ is finite or countable. In fact, the critical values can form a dense set (see Remark 6 in §5).

Since we can, by a local perturbation, change the value of a function at any specified critical point to any nearby value, it follows that the second condition for Morse functions cannot hold on a dense open subset of $\mathcal{F}^r$. However, given any pair of (non-degenerate) critical points for f, we can make a local perturbation at one of them that ensures these two belong to different level sets, and certainly every function near this new one also assigns them different values. Given f with no degenerate critical points, we can cover its critical set with disjoint discs centered at the critical points p_i, and find $\mathcal{N}_\varepsilon(f)$ so that each $g \in \mathcal{N}_\varepsilon(f)$ has a unique (non-degenerate) critical point q_i in B_i, and no others. The argument above shows that for each pair of distinct indices i, j the subset of $g \in \mathcal{N}_\varepsilon(f)$ for which q_i and q_j belong to different level sets is open and dense in $\mathcal{N}_\varepsilon(f)$. It follows from the Baire category theorem that the intersection of these sets (for all pairs $i \neq j$) is residual (and dense). We have shown the following.

LEMMA 2. *The set of Morse functions in the plane is a residual subset of $\mathcal{F}^r$ for $r \geq 2$.*

Now, the necessity of condition (i) and part of (ii) in Theorem 3 is a consequence of the following.

Remark 3. If $f \in \mathcal{F}$ is a Morse function, then its Hamiltonian flow Φ_f satisfies
(1) every equilibrium point is a hyperbolic saddle or a non-degenerate center;
(2) there are no saddle connections between distinct finite saddles.

The first statement follows from the local structure of the level sets of a function near a non-degenerate critical point (since the level curves of f are the trajectories of Φ_f), while for the second we observe that since a saddle connection for Φ_f is contained in a level curve of f, two finite saddles which are connected must belong to the same level set.

We close this section with some further dynamical properties of Hamiltonian flows in $\mathbb{R}^2$. Note that a transversal to the flow Φ_f (an embedded interval T nowhere tangent to the velocity X_f) has the further property that f is strictly monotone along T. Thus, a given transversal intersects any orbit in at most one point, so that an orbit intersecting the same transversal twice is periodic.

Recall that the *ω-limit set* (respectively, *α-limit set*) of $p \in \mathbb{R}^2$ under the flow Φ is the set $\omega(p)$ (respectively, $\alpha(p)$) of accumulation points of sequences of the form $\Phi^{t_i}(p)$ where $t_i \to \infty$ (respectively, $t_i \to -\infty$). In the Hamiltonian case, since f is invariant under the flow Φ_f, $\omega(p)$ and $\alpha(p)$ lie in the same level set of f as p, and so a regular point lies in $\omega(p)$ or $\alpha(p)$ only if it lies on the orbit of p, and that orbit is periodic. On the other hand, a non-degenerate critical point q lying in $\omega(p)$ (respectively, $\alpha(p)$), if $p \neq q$, is necessarily a saddle point with p on one of its stable (respectively, unstable) separatrices. Note, finally, that a semi-orbit escapes to infinity if and only if the appropriate limit set (ω or α) is empty. Thus we have the following.

Remark 4. Suppose Φ_f is a Hamiltonian flow satisfying both conditions of Remark 3. Then for each $p \in \mathbb{R}^2$, exactly one of the following holds:
(1) p escapes to infinity as $t \to \infty$, and $\omega(p) = \emptyset$;
(2) the orbit of p is periodic, and equals $\omega(p)$;
(3) p is an equilibrium point, and $\omega(p) = \{p\}$;
(4) p belongs to a stable separatrix of a saddle point q, and $\omega(p) = \{q\}$.
Also, exactly one of the analogous statements with $t \to \infty$ (respectively, $\omega(p)$) replaced by $t \to -\infty$ (respectively, $\alpha(p)$) and 'stable' with 'unstable' holds for p.

The following result relates the notion of virtual critical points for $f \in \mathcal{F}$ to saddles at infinity for the associated Hamiltonian flow Φ_f.

LEMMA 3. *Suppose p is a virtually critical point for the Morse function f. Then the level curve through p intersects a separatrix for some saddle at infinity of the associated Hamiltonian flow Φ_f.*

Proof. Suppose, as in Definition 3, C_i is a sequence of simple curves, with endpoints p_i, q_i, such that f takes the constant value $c_i \neq f(p)$ on C_i, $p_i \to p$, $q_i \to q$, and $r_i \in C_i$ is a sequence with no accumulation points. We can assume that p and q are regular points and that p_i (respectively, q_i) form a monotone sequence in some transversal T_p (respectively, T_q) through p (respectively, q).

We claim that C_i can be replaced by orbit segments of Φ_f. To this end it suffices to show that for each i, the quadrilateral Q_i bounded by C_{i-1}, C_{i+1} and the transversal segments $[p_{i-1}, p_{i+1}]$, $[q_{i-1}, q_{i+1}]$ contains an orbit segment of Φ_f which meets both transversals. Note that all points on the same transversal edge of Q_i enter Q_i in the same time direction. If some such point subsequently leaves Q_i, it does so via the other transversal edge. If a point x fails to leave Q_i, then by Remark 4 it lies on a separatrix of a saddle s in Q_i, and the whole segment of this separatrix from x to s is contained in Q_i. Since Q_i is compact, there are only finitely many possibilities for s and, for each, at most four points can fit the preceding description for x. It follows that all but finitely many points entering Q_i via one transversal leave it via the other, proving our claim.

Now, it follows that there exist $t_i > 0$ such that

$$C_i = \{\Phi^t_f(p_i) \mid 0 \leq t \leq t_i\}.$$

If a subsequence of $\{t_i\}$ converges, say to t, then $q = \Phi^t_f(p)$, and any sequence $r_i \in C_i$ has a subsequence converging to a point on the orbit segment from p to q. Hence we must have $t_i \to +\infty$, and $q \in \mathcal{J}_+(p)$. Observe that p (respectively, q) is not periodic, because otherwise every point on a transversal would be periodic with nearby period, and in particular the t_i would be bounded by the period of p_i, contradicting our last conclusion.

Finally, by Remark 4, if p (respectively, q) fails to escape to infinity in forward (respectively, backward) time, then it belongs to a stable (respectively, unstable) separatrix of some finite saddle, and for some point p' (respectively, q') on an unstable (respectively, stable) separatrix of the same saddle point—which does escape to infinity—we still have $q' \in \mathcal{J}_+(p')$, by Remark 2. Since p' belongs to the same level curve as p, the lemma follows. □

3. *Behavior at infinity*

In this section, we consider structures associated to saddles at infinity for a planar Hamiltonian flow Φ_f, and complete the proof that the conditions of Theorem 3 are necessary for Hamiltonian stability.

Fix a function $f \in \mathcal{F}^r$ with its associated Hamiltonian flow Φ_f and velocity vectorfield X_f. Observe that whenever $q \in \mathcal{J}_+(p)$ under Φ_f, we must have $f(q) = f(p)$. In particular, it makes sense to talk about the value of f at a saddle at infinity: any such value is a *virtual critical value* of f (equivalently, a virtual critical value is any value occuring at some virtually critical point); collectively, the critical and virtual critical values are the *extended critical values* of f. While the (ordinary) critical values can be dense in $\mathbb{R}$ for an open set of functions, we have seen that generically there are countably many. We wish to establish this also for the extended critical values. The argument is based on the following observations.

Remark 5. A homoclinic contour (finite or at infinity) for a Hamiltonian flow is a limit of closed orbits.

This follows immediately from the observation that, if a sequence $p_i \to p$ and $t_i \to \infty$ satisfy $q_i = \Phi^{t_i}(p_i) \to q = p$ with p a regular point, then the trajectories of the p_i must cross a transversal through p twice, and hence must be closed orbits.

LEMMA 4. *Suppose T is a transversal to Φ_f and p_i, $i = 1, 2, 3$ are distinct points at which T intersects the stable separatrices of some saddles at infinity. Then the unstable separatrices of these saddles cannot all intersect a single transversal T'.*

Proof. We will prove the lemma by contradiction. Number the p_i's so that $f(p_1) < f(p_2) < f(p_3)$, and suppose $p_i' \in T'$, $i = 1, 2, 3$, are the intersections of the corresponding unstable separatrices with a transversal T'. We can assume that T and T' are disjoint. We have for $i = 1, 2, 3$

$$\omega(p_i) = \alpha(p_i') = \emptyset$$
$$p_i' \in \mathcal{J}_+(p_i).$$

The latter says we can pick $q_i \in T$, $q_i' \in T'$, $i = 1, 2, 3$, with $q_i' = \Phi^{t_i}(q_i)$, $t_i > 0$, q_i arbitrarily near p_i, and q_i' arbitrarily near p_i'. We can assume that $\Phi^t(q_i) \notin T \cup T'$ for $0 < t < t_i$.

We claim also that p_2 and p_2' lie on different orbits. For otherwise, by Remark 5, q_2 can be taken to lie on a closed orbit, which must separate p_1 from p_3, and hence one of these two orbits must be bounded (in both time directions), a contradiction to the hypothesis that each escapes to infinity in some time direction. As a consequence, the orbit of p_2 cannot cross T'.

Let C_i denote the orbit segment from q_i to q_i', and for two points $\alpha, \beta \in T$ (respectively, $\alpha', \beta' \in T'$), $[\alpha, \beta]$ (respectively, $[\alpha', \beta']$) the segment of T (respectively, T') between them.

By the Jordan curve theorem, the union $[q_1, q_3] \cup C_3 \cup [q_3', q_1'] \cup C_1$ separates the plane into two components, one a topological disc and the other unbounded. Every point on the transversal $[q_1, q_3]$ leaves one of these components, say D_1, and enters the other, D_2. Since C_1 and C_3 are orbit segments, the only other possible passage between D_1 and D_2 is across $[q_1', q_3']$, where points pass from D_2 to D_1. Since $p_2 \in [q_1, q_3]$ has an unbounded forward semi-orbit which does not cross $[q_1', q_3']$, D_2 must be unbounded (and D_1 a disc). Since p_1 and p_3 also have unbounded forward semi-orbits, they must lie on $[q_1, q_3]$. Now, q_2, which is near p_2, must lie between p_1 and p_3. By an argument similar to that giving D_1 and D_2, the union $[q_2, q_3] \cup C_3 \cup [q_3', q_2'] \cup C_2$ separates the plane into two components, D_1' and D_2', with

$$D_1 \subset D_1', \quad D_2' \subset D_2,$$

and passage between D_1' and D_2' can occur only along the transversal segments $[q_2, q_3]$ and $[q_3', q_2']$. By looking at the values of f, we see that the forward semi-orbit of p_1 is contained in D_1', while that of p_3 is contained in $D_2' \cup D_1$. Since one of these two sets is bounded, the semi-orbits of p_1 and p_3 cannot both escape to infinity, a contradiction. □

As a corollary of Lemma 4 we obtain our desired counting result.

PROPOSITION 1. *A planar function $f : \mathbb{R}^2 \to \mathbb{R}$ has at most countably many virtual critical values.*

Proof. We can cover the set of regular points of f with countably many *flowboxes*: sets of the form $\{\Phi^t(x) \mid x \in T, 0 \le t \le \tau\}$ where T is a transversal and $0 < \tau < \infty$. For each saddle at infinity, we pick a disjoint pair of these flow boxes such that the first (respectively, second) intersects the stable (respectively, unstable) separatrix of the saddle. By Lemma 4, a particular pair of choices can be associated to at most two different saddles at infinity; since there are countably many possible pairs, this proves the proposition. □

Using this result, we can show that the rest of the conditions in Theorem 3 are necessary for Hamiltonian stability.

PROPOSITION 2. *If f is Hamiltonian $\mathcal{C}^r$ stable ($r \ge 1$) then every critical curve for f is isolated.*

Proof. Since the Morse functions are dense, we can assume f is Morse, so that q is the only critical point of f with value equal to $c = f(q) = f(p)$. Since non-degenerate local extrema are automatically isolated from other critical and virtually critical points, we can assume q is a saddle point.

Let U be a neighborhood of q containing no other critical points of f, bounded by transversals to the four separatrices of q and orbit segments joining their ends. We can assume that one of these transversals, T, goes through p. Let V and W be closed neighborhoods of q such that

$$V \subset \operatorname{int}(W), \quad W \subset \operatorname{int}(U).$$

Consider perturbations g of f of the form

$$g(x, y) = f(x, y) + \delta(x, y)$$

where $\delta : \mathbb{R}^2 \to \mathbb{R}$ is a 'bump function' satisfying, for some constant $\delta_0 > 0$,

$$\begin{aligned} \delta = 0 &\quad \text{off } W \\ 0 \le \delta \le \delta_0 &\quad \text{on } W \\ \delta = \delta_0 &\quad \text{on } V. \end{aligned}$$

Given a $\mathcal{C}^r$ neighborhood $\mathcal{N}_\varepsilon(f)$ of f, it is possible to find $\alpha_0 > 0$ such that, for every value of δ_0 satisfying $0 < \delta_0 < \alpha_0$, there exists δ as above such that $g \in \mathcal{N}_\varepsilon(f)$. Every such perturbation has q a hyperbolic saddle for Φ_g, with $g(q) = c + \delta_0$, and by narrowing $\mathcal{N}_\varepsilon(f)$ we can ensure that g has no other critical point in U. The stable and unstable separatrices of q leave U at the points on the four transversals where $f = c + \delta_0$, and the vectorfields X_f and X_g agree off U.

It follows that if $c + \delta_0$ is not an extended critical value of f, then Φ_g has no saddle connection between q and another finite or infinite saddle. By Proposition 1, the complement of the extended critical values of f is dense in $\mathbb{R}$. A dynamical equivalence between Φ_f and Φ_g in an appropriate compact-open neighborhood of the identity must take q to itself, and so Φ_f cannot be Hamiltonian stable unless f also has no connection between q and any other saddle.

Thus, if p is not isolated from stable and unstable separatrices of other saddles, then there exist $c_i \to c$ such that the Φ_f-orbit crossing T where $f = c_i$ is a stable or unstable separatrix of some saddle. Suppose $c_i > c$ (otherwise we define g as $f - \delta$). By condition (i) in Remark 2, we can, without loss of generality, assume that p lies on an unstable separatrix and all the orbits with $f = c_i$ are stable separatrices. A perturbation g as above with $c + \delta_0 = c_i$ then has a saddle connection between q and another saddle. Thus Φ_f and Φ_g cannot be dynamically equivalent (at least by a homeomorphism in some compact-open neighborhood of the identity), again preventing structural stability of Φ_f in $\mathcal{H}$. □

Combining Proposition 2 with Lemma 2 and Remark 3, we have shown necessity in Theorem 3. We summarize.

PROPOSITION 3. *If f (respectively, Φ_f) is Hamiltonian $\mathcal{C}^r$ stable for some $r \ge 1$, then*

(1) *every critical point of f is non-degenerate (i.e. every equilibrium of Φ_f is a hyperbolic saddle or non-degenerate center);*

(2) *each critical curve for f is proper (i.e. the separarices of each finite saddle are isolated from the separatrices of all other finite or infinite saddles for Φ_f).*

4. *Stability results*

In this section, we will show that the conditions of Theorem 3 are sufficient for Hamiltonian stability.

Fix $f \in \mathcal{F}^2$ satisfying

(1) every critical point of f is non-degenerate, and

(2) each critical curve is proper.

Also, fix $K \subset \mathbb{R}^2$ compact and $\delta > 0$. We shall prove f is Hamiltonian $\mathcal{C}^1$ stable by defining a strong $\mathcal{C}^2$ neighborhood $\mathcal{N}_\varepsilon(f)$ and, for each $g \in \mathcal{N}_\varepsilon(f)$, a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ mapping level curves of f to level curves of g, with $|h(x) - x| < \delta$ for all $x \in K$. Our construction of h is greatly influenced by the proof of [**dPW**, Theorem 3.6.1].

We will make extensive use of the gradient lines of f. Recall that the gradient vectorfield $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ vanishes precisely at the critical points of f and is orthogonal at each regular point to the Hamiltonian vectorfield X_f. The gradient flow, Γ_f, generated by ∇f (whose trajectory through a point p, denoted γ_p, will be called the *gradient line* of f through p) satisfies:

(1) Γ_f has a node wherever Φ_f has a center: a local maximum (respectively, minimum) for f is a sink (respectively, source) for Γ_f;

(2) Γ_f has a hyperbolic saddle wherever Φ_f does (i.e. at every non-degenerate non-extremum critical point of f);

(3) the gradient lines foliate the set of regular points of f transversally to the flow lines of Φ_f (i.e. the level curves of f).

By condition (3), the function f is strictly monotone along each gradient line, and so for each regular point $x \in \mathbb{R}^2 \setminus \mathrm{crit}(f)$, the set of values achieved by f along the gradient line through x forms an open interval

$$I(x) = \{f(\Gamma_f^t(x)) \mid t \in \mathbb{R}\} = (m_-(x), m_+(x))$$

where

$$m_\pm(x) = \lim_{t\to\pm\infty} f(\Gamma_f^t(x)).$$

Define $l_\pm(x)$ by

$$m_\pm(x) = f(x) \pm l_\pm(x)$$

so that $l_\pm(x)$ is strictly positive, possibly infinite. Given $s \in I(x)$, we can find $t \in \mathbb{R}$ with $s = f(\Gamma_f^t(x))$, and if $x_i \to x$, then $s_i = f(\Gamma_f^t(x_i)) \to s$, with $s_i \in I(x_i)$. It follows that each of the two functions $l_\pm(x)$ is lower semicontinuous. Thus we can find arbitrarily smooth functions $\mu_\pm(x)$ defined on the regular points such that

$$0 < \mu_\pm(x) < l_\pm(x)$$

for every $x \in \mathbb{R}^2 \setminus \mathrm{crit}(f)$. Given a pair of such functions and a subset $A \subset \mathbb{R}^2 \setminus \mathrm{crit}(f)$ whose intersection with each level curve is relatively open, we can form a neighborhood of A via

$$\mathcal{U}(A, \mu_\pm) = \{y = \Gamma_f^t x \mid x \in A \text{ and } f(x) - \mu_-(x) < f(y) < f(x) + \mu_+(x)\}.$$

The gradient lines foliate $\mathcal{U}(A, \mu_\pm)$ in such a way that for each $x \in A$, the function f takes each value in the interval $(f(x) - \mu_-(x), f(x) + \mu_+(x))$ at a unique point on the leaf Γ_x through x.

We shall combine a structure of this sort with a system of open sets $V'(p) \subset V(p)$, $p \in \mathrm{crit}(f)$, where the $V(p)$'s are disjoint, such that for each $p \in \mathrm{crit}(f)$:

(1) $V(p)$ is the component of the preimage $f^{-1}\{(c_-, c_+)\}$ containing p, where $c_\pm = c_\pm(p)$ satisfy $c_- < f(p) < c_+$, and the critical curve through p is the only critical or virtually critical curve intersecting $V(p)$;

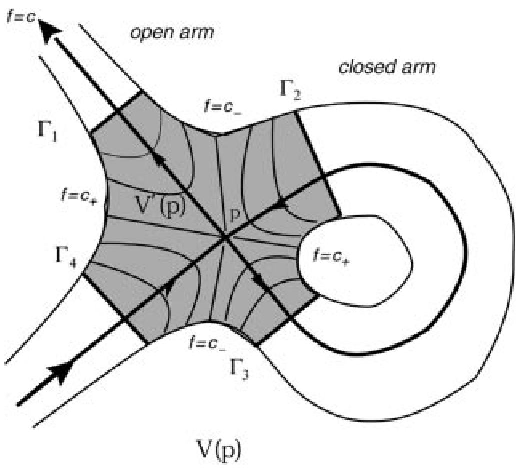

FIGURE 5. $V(p)$ and $V'(p)$ for a saddle point.

(2) $V'(p)$ is a neighborhood of p with diameter at most $\delta/2$, and for each $x \in V'(p) \setminus \{p\}$,

$$\Gamma_x \cap V(p) = \Gamma_x \cap V'(p)$$

is an interval with

$$f(\Gamma_x \cap V(p)) = J(x) = (a, b),$$

where $a, b \in \{c_-, c_+, f(p)\}$ (the possibilities depend on whether p is a maximum, minimum or saddle for f).

When p is a local maximum (respectively, minimum) for f and $c = f(p)$, we can find $c_-(p) < c$ (respectively, $c_+(p) > c$) sufficiently close to c that the level curve where $f = c_-$ (respectively, c_+) bounds a topological disc of diameter at most $\delta/2$ containing p and filled by simple closed level curves surrounding p with $f = s$ for each $s \in (c_-, c)$ (respectively, (c, c_+)). Picking $c_- < c$ (respectively, $c_+ > c$) arbitrarily, we obtain $V'(p) = V(p)$ satisfying both conditions.

When p is a saddle point, we can still invoke the second condition on f to pick $c_- < f(p) < c_+$ so that the component $V(p)$ of $f^{-1}\{(c_-, c_+)\}$ containing p intersects no other critical or virtually critical curves of f, and so that each Γ_f-separatrix at p reaches one of the level sets $\mathcal{L}_{c_\pm}(f)$. By taking $c_\pm$ sufficiently close to each other, we can make each of these Γ_f-separatrix segments arbitrarily short, and then define $V'(p)$ to be the union of the gradient line segments $\Gamma_x \cap V(p)$ which intersect some small neighborhood of p: this can be made to have arbitrarily small diameter, as well (see Figure 5).

Observe that the relative boundary of $V'(p)$ in $V(p)$ consists of four gradient lines Γ_i, $i = 1, \ldots, 4$, each crossing the critical curve of f through p in a unique point. We will refer to the components of $V(p) \setminus \bigcup_{i=1}^{4} \Gamma_i$ other than $V'(p)$ as *arms* of $V(p)$. Each arm intersects a unique separatrix orbit for p under the Hamiltonian flow Φ_f. If this separatrix is a homoclinic contour, the arm is a relatively compact rectangle bounded by two gradient lines Γ_{i_1}, Γ_{i_2} and a finite segment of each of the level curves $\mathcal{L}_{c_\pm}(f)$ bounding $V(p)$; we will call this a *closed arm*. By contrast, a separatrix which escapes to infinity is trapped in an *open arm* of $V(p)$, bounded by one gradient line Γ_i and an unbounded segment $\mathcal{L}_\pm$ of each of the level curves $\mathcal{L}_{c_\pm}(f)$. *A priori*, the boundary of an open arm may also contain additional components of either of the level sets $\mathcal{L}_{c_\pm}(f)$, but the second condition on f prevents this, as follows easily from the next lemma.

LEMMA 5. *Suppose* Γ *is an open interval with endpoints* q_-, q_+, *transverse to* Φ. *Suppose a saddle* p *has an unstable (respectively, stable) separatrix which escapes to infinity, crossing* Γ, *and* $\operatorname{clos}\Gamma$ *intersects no stable (respectively, unstable) separatrices of other finite or infinite saddles. Then the forward (respectively, backward) saturation of* Γ

$$S_\pm = \{\Phi^t(q) \mid q \in \Gamma, \pm t > 0\}$$

is an open set whose boundary in $\mathbb{R}^2$ *is the union of* Γ *with the forward (respectively, backward) orbits* $\mathcal{L}_\pm = \{\Phi^t(q_\pm) \mid t \geq 0\}$ *(respectively,* $\mathcal{L}_\pm = \{\Phi^t(q_\pm) \mid t \leq 0\}$*) of the endpoints of* Γ.

Proof. We prove the case of an unstable separatrix for p; the stable case follows by time reversal. Note that, by Remark 4, our hypotheses on Γ imply that $\omega(q) = \emptyset$ for each $q \in \operatorname{clos}\Gamma$.

Suppose r is a boundary point of S_+ which does not lie on $\Gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-$. Pick $r_i \to r$ with $r_i \in S_+$, so $r_i = \Phi^{t_i}(q_i)$ for some $q_i \in \Gamma$ and $t_i > 0$. Going to a subsequence, $q_i \to q \in \operatorname{clos}\Gamma$. If some subsequence of t_i converges, say to t, then $r = \Phi^t(q)$, and $r \in S_+ \cup \Gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-$. But this contradicts the hypothesis that r is a boundary point of S_+ not in $\Gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-$, so $t_i \to +\infty$. However, then q belongs to the stable separatrix of some saddle at infinity, contradicting our assumption on Γ. □

We now wish to foliate the set of regular points by curves T transverse to X_f so that, whenever $x \in V(p) \setminus \{p\}$ for some critical point p, the curve T_x through x intersects the boundary of $V(p)$ in at least one point $\sigma_\pm(x) \in \mathcal{L}_{c_\pm}(f)$, and if not at two, then the other end of T_x is p. A first candidate for this foliation is the gradient lines: indeed, the choice $T_x = \Gamma_x$ satisfies this condition if $x \in V'(p) \setminus \{p\}$ (and thus for $x \in V(p) \setminus \{p\}$ when p is a local extremum for f). Similarly, if x belongs to a closed arm of a saddle point p, then the gradient flow enters this arm along $\mathcal{L}_{c_-}(f)$, has no ω-limit inside the arm (since there are no equilibria), and hence must leave the arm—which is only possible via $\mathcal{L}_{c_+}(f)$. It follows that we can take T_x tangent to ∇f in $V'(p)$ for each $p \in \operatorname{crit}(f)$, and in every closed arm of $V(p)$ for each saddle.

However, we have no guarantee that this condition holds in an open arm, since we cannot exclude the possibility of saddles at infinity for the *gradient* flow. We will need, therefore, to define T as a modification of the gradient foliation inside (some) open arms of saddles. To make sure our construction yields a global foliation (particularly at points of accumulation of arms for different saddles), we need to control it using the sets $\mathcal{U}(A, \mu_\pm)$ described earlier.

Denote by $\mathcal{V}$ the union of all open arms of saddles; the complement $\mathcal{V}^{\mathrm{c}}$ is a closed set containing all the sets $V'(p)$ (for *all* critical points) as well as the bounding level curves $\mathcal{L}_{c_\pm}(f)$ for all sets $V(p)$. We would like to pick smooth functions $\mu_\pm$ on $\mathcal{V}^{\mathrm{c}}$ so that the set

$$\mathcal{U} = \mathcal{U}(\mathcal{V}^{\mathrm{c}} \setminus \operatorname{crit}(f), \mu_\pm) \cup \operatorname{crit}(f)$$

is an open set containing $\mathcal{V}^c$. This is not automatic, because $\mathcal{V}^c$ need not intersect each level curve in a relatively open set. However, the only points which are not automatically interior to $\mathcal{U}$ (for *every* positive choice of $\mu_\pm$) are those along the gradient lines Γ_i bounding the open arms in $\mathcal{V}$. Clearly, we can pick $\mu_\pm$ so that near the ends of Γ_i the gradient line segments defining $\mathcal{U}$ join $\mathcal{L}_-$ to $\mathcal{L}_+$, and this ensures that Γ_i is interior to $\mathcal{U}$.

A boundary point z of $\mathcal{U}$ must belong to an open arm of $V(p)$ for some saddle p, and so the intersection with $V(p)$ of some gradient line must contain an open interval with one endpoint at z and the other at a boundary point $\sigma_\pm(z) \in \mathcal{L}_{c_\pm}(f)$ of $V(p)$. If both possibilities ($\sigma_+(z)$ and $\sigma_-(z)$) occur, then the gradient line through z intersects $V(p)$ in an open interval joining $\mathcal{L}_{c_-}(f)$ to $\mathcal{L}_{c_+}(f)$, with z the only point of this interval not in $\mathcal{U}$.

It will prove useful to introduce a certain coordinate system on each open arm $V_i(p)$.

LEMMA 6. *For each open arm $V_i(p)$, there is a diffeomorphism of* $\operatorname{clos} V_i(p)$ *with the subset of* $\mathbb{R}^2$

$$\tilde{V}_i(p) = [c_-, c_+] \times [0, \infty)$$

under which Γ_i corresponds to $(c_-, c_+) \times \{0\}$ and, for each $s \in [c_-, c_+]$, $\mathcal{L}_s(f) \cap V_i(p)$ corresponds to $\{s\} \times [0, \infty)$, and which carries X_f to the unit vertical vectorfield.

Proof. The transports under Φ^t_f of the bounding transversal Γ_i of $V_i(p)$ foliate its saturation $S_\pm$, which by Lemma 5 equals $V_i(p)$, but then the inverse of the desired diffeomorphism is given by $(s, t) \mapsto \Phi^t_f(q_s)$, where q_s is the unique point on $\operatorname{clos}\Gamma_i$ with $f(q_s) = s$. ☐

Since the gradient lines in $V_i(p)$ are transverse to X_f, they correspond to the graphs $y = \gamma(x)$ of functions defined on subintervals of $[c_-, c_+]$. We would also like to have the boundary of $\mathcal{U}$ in $V_i(p)$ described by a function $y = u(x)$.

Note that if a gradient line in $V_i(p)$ joins both sides (i.e. it corresponds in $\tilde{V}_i(p)$ to the graph of a function $y = \gamma(x)$ defined on all of $[c_-, c_+]$), then the same holds for the gradient line passing through any point (x, y) with $y < \gamma(x)$. Thus, if a gradient line joining the sides of $V_i(p)$ meets the boundary of $\mathcal{U}$ in $V_i(p)$ at a unique point, then we can 'thicken' $\mathcal{U}$ to engulf all points below this gradient line. In this way, we ensure that a gradient line in $V_i(p)$ above some unique point z intersects the complement of $\mathcal{U}$ in an open subinterval. Finally, by making the functions $\mu_+(x)$ on $\mathcal{L}_{c_-}$ and $\mu_-(x)$ on $\mathcal{L}_{c_+}$ strictly decreasing ($\mu_\pm(\Phi^t_f(x)) < \mu_\pm(x)$ for $x \in \mathcal{L}_{c_\mp}$ and $t > 0$), we can ensure that the boundary of $\mathcal{U}$ in $V_i(p)$ corresponds to a graph $y = u(x)$ in $\tilde{V}_i(p)$, where u is defined on some open interval $(a_-, a_+) \subset (c_-, c_+)$, with a unique minimum at z and strictly monotone on either side, and $\lim_{x \to a_\pm} u(x) = +\infty$. Let $z = (x_0, y_0)$.

With this picture, we see that for each $r > y_0$, there are precisely two points (x_-, r) and (x_+, r) with

$$c_- \le a_- < x_- < x_0 < x_+ < a_+ \le c_+$$

and

$$u(x_\pm) = r.$$

By definition, there is a gradient line in $\mathcal{U}$ joining a point of $\mathcal{L}_{c_\pm}$ to $(x_\pm, r)$, and so we can form a foliation of $V_i(p)$ by piecewise-smooth curves transverse to the level sets of

f which agree with the gradient lines in $\mathcal{U}$ and correspond to horizontal line segments in $V_i(p)\setminus\mathcal{U}$. The possible 'corners' along $y = u(x)$ are easily smoothed out locally to obtain a smooth foliation T of $V_i(p)$ such that

(1) T_x is tangent to ∇f if $x \in \mathcal{U}$, and transverse to X_f everywhere;

(2) each leaf crosses both $\mathcal{L}_{c_-}(f)$ and $\mathcal{L}_{c_+}(f)$.

We would also like to ensure that the leaves in $\mathcal{V}$ intersecting our compact set K have length at most $\delta/2$. This is guaranteed inside $V'(p)$, and can be guaranteed inside any closed arm by controlling $|c_+ - c_-|$. Similarly, we can uniformly estimate the derivatives on K of the diffeomorphism taking the arm $V_i(p)$ to the strip $\tilde{V}_i(p)$, and thus (reducing $|c_+ - c_-|$ if necessary) ensure that our construction creates no long leaves through K.

Since the gradient lines already foliate $\mathcal{U}\setminus\text{crit}(f)$ (and the critical points form a discrete set), we have the following.

LEMMA 7. *Given $f \in \mathcal{F}^2$ satisfying*

(1) *every critical point is non-degenerate,*

(2) *each critical curve is proper,*

as well as a compact set $K \subset \mathbb{R}^2$ and $\delta > 0$, there exist open sets $V'(p) \subset V(p)$, $p \in \text{crit}(f)$, pairs of numbers $c_\pm = c_\pm(p)$, $c_- < f(p) < c_+$, a foliation T of $\mathbb{R}^2\setminus\text{crit}(f)$ by curves transverse to X_f and an open cover $\{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{R}^2$ such that:

(1) *$V(p)$ is the component of $f^{-1}\{(c_-, c_+)\}$ containing p, and contains no other critical or virtually critical points of f;*

(2) *$V'(p)$ is a neighborhood of p contained in $\mathcal{U}\setminus\mathcal{V}$ of diameter at most $\delta/2$; $V'(p) = V(p)$ if p is a local extremum, and otherwise is cut out of $V(p)$ by gradient lines Γ_i crossing each Φ_f-separatrix for p;*

(3) *T_x is tangent to ∇f for $x \in \mathcal{U}$;*

(4) *if $x \in V(p)\setminus\{p\}$, then T_x (the leaf through x) crosses at least one of the bounding level sets $\mathcal{L}_{c_\pm}(f)$, and if it does not cross both, then p is an endpoint of T_x; for $x \in K \cap V(p)\setminus\{p\}$, the set $T_x \cap V(p)$ has diameter at most $\delta/2$.*

With this structure, we can prove sufficiency.

PROPOSITION 4. *Suppose $f \in \mathcal{F}^2$ satisfies*

(1) *every critical point is non-degenerate,*

(2) *each critical curve is proper.*

Given $K \subset \mathbb{R}^2$ compact and $\delta > 0$, there exists $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^+$ such that, for every $g \in \mathcal{N}_\varepsilon(f)$, we can find a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking level curves of f to level curves of g, with $|h(x) - x| < \delta$ whenever $x \in K$.

Proof. We construct the sets $V'(p) \subset V(p)$ and the foliation T as in Lemma 7. Now, pick for each $p \in \text{crit}(f)$ a pair of numbers $0 < r_0 < r_1$ so that the closed disc $W_i(p)$ of radius r_i is interior to $V'(p)$, and let

$$\mathcal{W}_i = \bigcup_{p\in\text{crit}(f)} W_i(p), \quad i = 0, 1.$$

Let $\pi : \mathbb{R}^2 \to [0, 1]$ be a smooth function such that $\pi = 1$ on $\mathcal{W}_0$ and $\pi = 0$ off $\mathcal{W}_1$.

Given $g \in \mathcal{F}^2$, we will 'localize' g by defining

$$g_1(x) = f(x) + \pi(x)[g(x) - f(x)].$$

Clearly, we have

$$\begin{aligned} g_1(x) &= g(x) \quad \text{on } \mathcal{W}_0 \\ g_1(x) &= f(x) \quad \text{off } \mathcal{W}_1. \end{aligned}$$

As a notational convenience, set

$$g_0(x) = f(x), \quad g_2(x) = g(x).$$

Note that the value and derivatives of π and f at a point determine a 'local Lipschitz constant' $\lambda(x) > 0$ such that

$$\|g_j(x) - g_{j-1}(x)\|_{\mathcal{C}^2} < \lambda(x)\|g(x) - f(x)\|_{\mathcal{C}^2}$$

so that by picking an appropriate function $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^+$ we can guarantee any set of *a priori* pointwise estimates on $\|g_j(x) - g_{j-1}(x)\|_{\mathcal{C}^2}$ for $j = 1$ and $j = 2$ via the condition $g \in \mathcal{N}_\varepsilon(f)$. We will therefore concentrate on our pointwise estimates, defining ε implicitly.

We will require the following.

(1) For each $p \in \mathrm{crit}(f)$, g_1 has a unique critical point q in $W_1(p)$, and it belongs to $W_0(p)$ (and is of the same type as p).

(2) For $x \in \mathbb{R}^2 \setminus \mathcal{W}_0$, $g_j(x) \in I(x)$, and for $x \in W_1(p)$,

$$|g_j(x) - f(x)| < \tfrac{1}{3} \max\{f(x) - c_-, c_+ - f(x)\}.$$

(3) Let K_0 be a compact set containing $K \setminus \mathcal{W}_0$ in its interior and disjoint from the critical points of f; we want $|g_j(x) - g_{j'}(x)| < \varepsilon'/2$ for all $x \in K_0$, where $\varepsilon' > 0$ is determined so that for $y \in K_0$, x within distance ε' of y and z on the same leaf T_x as x, the estimate $|f(x) - f(z)| < \varepsilon'$ implies $|x - z| < \delta/2$.

The first of these is guaranteed by $\mathcal{C}^2$ estimates on $\mathcal{W}_1$ (since $g_0 = f$ off $\mathcal{W}_1$) and the second are $\mathcal{C}^1$ estimates. To see that ε' in the third condition exists, note that $\varepsilon'(y) > 0$ depending on the regular point y exists by the transversality of T to the level sets of f. On K_0, this can be bounded away from zero, and hence replaced by a constant.

Given $g \in \mathcal{F}^2$ satisfying these conditions, we will construct our homeomorphism in two stages: h_j ($j = 0, 1$) will take level curves of g_j to level curves of g_{j+1}, and $h = h_1 \circ h_0$. Note that the triangle inequality gives $h \in \mathcal{N}_{\mathrm{co}}(K, U)$ provided $|h_j(x) - x| < \delta/2$ for all $x \in K$ and $j = 0, 1$.

To construct h_0, we first define, for each $p \in \mathrm{crit}(f)$, a strictly increasing function $\varphi_p : \mathbb{R} \to \mathbb{R}$ which differs from the identity only on $f(W_1(p))$ and takes $c = f(p)$ to $c' = g_1(q)$. We also form a new foliation, T', analogous to the foliation T of Lemma 7, but using the gradient flow of g_1 in place of that for f. Note that by hypothesis, the two foliations agree (i.e. local leaves agree) off $\mathcal{W}_1$. We need, however, to take care with the exceptional leaves formed by the separatrices of the gradient flows at saddles. Suppose p is a saddle point for f, and q the unique critical point of g_1 in $W_1(p)$. Even though the two foliations have the same *local* leaves near the boundary of $V'(p)$, a ∇f-separatrix for

p and the corresponding ∇g_1-separatrix for q may meet this boundary at different points. However, with a non-linear shear near each of the bounding level curve segments $\mathcal{L}_\pm$, we can modify the foliation T' for g_1 slightly inside the boundaries $\mathcal{L}_\pm$ of $V'(p)$ so that the leaf of T' which becomes a ∇g_1-separatrix for q inside $W_0(p)$ meets $\mathcal{L}_\pm$ at the same point as the corresponding ∇f-separatrix for p. With a little care, we can also ensure that if T_x is a non-separatrix leaf (of the foliation T for f), with endpoints $\sigma_\pm(x)$, then the modified foliation T' for g satisfies

$$T'_{\sigma_+(x)} = T'_{\sigma_-(x)}.$$

Now define $h_p(x)$, for $x \in V(p) \setminus \{p\}$ ($p \in \operatorname{crit}(f)$) to be the unique point on $T'_{\sigma_\pm(x)}$ where

$$g_1(h_p(x)) = \varphi_p(f(x)),$$

and $h_p(p) = q$. This defines a homeomorphism $h_p : V(p) \to V(p)$ which takes level curves of f in $V(p)$ to level curves of g_1 in $V(p)$, and which equals the identity near the bounding level curves of $V(p)$. Define

$$h_0(x) = \begin{cases} h_p(x), & x \in V(p), \text{ some } p \in \operatorname{crit}(f) \\ x, & \text{otherwise.} \end{cases}$$

Since $h_p = \mathrm{id}$ near the boundary of $V(p)$, the only possible problem with this definition is continuity at limit points $x = \lim x_i$, where $x_i \in V(p_i)$ with p_i distinct critical points. Since $\operatorname{crit}(f)$ is a discrete set, this means x is a regular point

$$x \in \mathcal{U} \setminus \bigcup_{p \in \operatorname{crit}(f)} V'(p) = \mathcal{U}',$$

so eventually $x_i \in \mathcal{U}'$. Now, in $\mathcal{U}'$ the leaves of T' agree with those of T, and so for each x_i, $h(x_i) \in T'_{x_i}$. These leaves converge to $T_x = \lim T_{x_i} = \lim T'_{x_i}$. Furthermore, since $V(p_i)$ intersects T_x in the set where f takes values in $(c_-(p_i), c_+(p_i))$, and these are disjoint, we must have

$$f(x) = \lim c_-(p_i) = \lim c_+(p_i) = \lim \varphi_{p_i}(f(x_i)).$$

It follows that $h_0(x_i) \to x = h_0(x)$, so that the homeomorphism h_0 is well defined. It is also clearly a homeomorphism taking level curves of $g_0 = f$ to level curves of g_1, and $|h_0(x) - x| < \delta/2$ for $x \in K$.

Now define h_1 off $\mathcal{W}_0$ to be the unique point on T'_x where

$$g(h_1(x)) = g_1(x).$$

This is well defined by the second condition on the perturbation, and gives $h_1(x) = x$ on the boundary of each $W_0(p)$. However, inside $W_0(p)$, g_1 and g agree, so this extends inside as the identity. Since the $W_0(p)$'s do not accumulate anywhere, h_1 is a well defined homeomorphism taking g_1-level curves to g_2-level curves. Finally, the third condition guarantees that $|f(h_1(x)) - f(x)| < \varepsilon'$, hence $|h_1(x) - x| < \delta/2$ for $x \in K \setminus \mathcal{W}_0 \subset K_0$.

In view of our earlier observations, this proves the proposition. $\square$

5. *Hamiltonian stability versus topological stability for planar functions*

In this section, we explore the distinction between our notion of Hamiltonian stability, based on the dynamics of the flows Φ_f, and topological stability, as in [**dPW**]. To distinguish these, we call $f, g : \mathbb{R}^2 \to \mathbb{R}$ *dynamically equivalent* if there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking level curves of f to level curves of g, and *functionally equivalent* if there exist homeomorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$g \circ h = \phi \circ f,$$

i.e. h takes level *sets* of f to level *sets* of g. A function $f \in \mathcal{F}^r$ is Hamiltonian (respectively, topologically) $\mathcal{C}^r$ stable if for some strong $\mathcal{C}^r$ neighborhood $\mathcal{N}_\varepsilon(f)$, every $g \in \mathcal{N}_\varepsilon(f)$ is dynamically (respectively, functionally) equivalent to f, with h in some pre-assigned compact-open neighborhood of the identity.

Clearly, functional equivalence implies dynamic equivalence, so that topological stability implies Hamiltonian stability. We shall show the converse fails on a non-empty open subset of $\mathcal{F}^r$ ($r \geq 1$). As a preliminary, we prove the following, which justifies a parenthetic assertion in §2.

LEMMA 8. *Suppose $f \in \mathcal{F}^2$ has a discrete closed set $C \subset \operatorname{crit}(f)$ of non-degenerate critical points whose values $f(C)$ are dense in some non-degenerate interval $[a, b]$ ($a < b$). Then the same is true of every function in some strong $\mathcal{C}^2$ neighborhood of f.*

Proof. Pick a countable family of open sets A_i, $i \in \mathbb{N}$, containing a basis for the neighborhoods of every number in $[a, b]$. Note that $f(C)$ must be countably infinite, and so we can pick a sequence p_i of distinct elements of C with $f(p_i) \in A_i$, $i = 1, \ldots$.

Since C is discrete, we can find disjoint neighborhoods U_i of p_i with $f(U_i) \subset A_i$, and p_i the only critical point in U_i. For each U_i, pick $\varepsilon_i > 0$ such that, whenever

$$\|g(x) - f(x)\|_{\mathcal{C}^2} < \varepsilon_i, \quad \forall x \in U_i$$

g has a unique critical point q_i in U_i, and $g(q_i) \in A_i$. Now, take $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^+$ with

$$\varepsilon(x) < \varepsilon_i, \qquad \text{for } x \in U_i.$$

Then $g \in \mathcal{N}_\varepsilon(f)$ implies $g(q_i)$ is a critical value in A_i, and hence

$$[a, b] \subset \operatorname{clos}\{g(q_i)\}$$

as required. □

Using this, one can prove easily that there exists a non-empty open subset of $\mathcal{F}^2$ in which every element has critical values dense in $\mathbb{R}$: for example, adding a small bump function around each $p_{ij} = (i, j)$ to the function $f_0(x, y) = x/2^y$ we can construct a function f with $C = \mathbb{Z} \times \mathbb{Z}$ and $f(C) = \{i/2^j\}$. We shall use Lemma 8 in a slightly different way.

Remark 6. If $f \in \mathcal{F}$ has a critical value which is a limit of other critical values, then it is not topologically $\mathcal{C}^r$ stable for any r.

To see this, note that we can assume all critical points are non-degenerate, hence $\mathrm{crit}(f)$ is discrete. If p_i, $i = 0, \ldots$ are critical points with $f(p_i) = c_i$ distinct and $c_i \to c_0$, then via a small bump function we can change f only very near p_0 so as to achieve $g(q_0) = c_i$ for some large i, and alternatively so as to have q_0 the only critical point in its level set. However, then the number of critical points in the level set through q_0, which is an invariant of functional equivalence, is not constant in a strong $\mathcal{C}^r$ neighborhood of f, preventing topological stability.

Using these observations, we can construct our example.

PROPOSITION 5. *There exists a non-empty open subset of $\mathcal{F}^r$, $r \geq 1$, in which every function is Hamiltonian stable, but not topologically stable.*

Proof. Consider first the function

$$u_0(x) = e^x \cos x$$

with

$$u_0'(x) = e^x(\cos x - \sin x)$$

and hence a relative maximum (respectively, minimum) at x_{2k} (respectively, x_{2k+1}) for each integer k, where

$$x_n = \frac{4n+1}{4}\pi.$$

Note that

$$u_0(x_n) = \frac{(-1)^n}{\sqrt{2}} e^{x_n}$$

and e^{x_n} grows monotonically with increasing n.

Now, the function

$$f_0(x, y) = u_0(x) + y^2$$

has a saddle point (respectively, local minimum) at $p_n = (x_n, 0)$ for n even (respectively, odd). The level curve through p_{2k} consists of a homoclinic contour through $q_{2k} = (x'_{2k}, 0)$, where

$$x_{2k+1} < x'_{2k} < x_{2k+2}, \quad u_0(x'_{2k}) = u_0(x_{2k})$$

and two separatrices escaping to infinity, with x-coordinate going to $-\infty$ and y-coordinate going to $+\infty$ (respectively, $-\infty$). The inside of the homoclinic contour is filled with closed level curves surrounding p_{2k+1}, while the level curves crossing the x-axis between q_{2k} and p_{2k+2} march off to infinity in parallelizable fashion. It is easy to verify that there are no saddles at infinity.

Now, we consider a modification u of u_0. Note that for $n \geq 0$, $|u_0(x_n)| > 1$. For each $k = 0, \ldots$, let a_k (respectively, b_k) be the unique point between x_{2k} and x_{2k+1} at which

$$u_0(a_k) = 1 \text{ (respectively, } u_0(b_k) = 0)$$

and modify u_0 on $[a_k, b_k]$ so that there are 2×3^k non-degenerate critical points in (a_k, b_k), with (distinct) critical values

$$c_{i,k} = \frac{i}{3^k} + \frac{1}{3^{k+1}}, \quad c'_{i,k} = \frac{i}{3^k} + \frac{2}{3^{k+1}}.$$

Note that the numbers $c_{i,k}$, $c'_{i,k}$ are all distinct, and their union is dense in $[0, 1]$.

Now consider

$$f(x, y) = u(x) + y^2.$$

We see that the new maxima (respectively, minima) of u yield saddles (respectively, minima) of f, and that the critical curves corresponding to the new saddles in $[a_k, b_k] \times \{0\}$ are pairs of homoclinic contours, contained inside the homoclinic contour through p_{2k}. This, together with the distinctness of the critical values, shows that f satisfies the hypotheses of Theorem 3, and hence an open strong $\mathcal{C}^r$ neighborhood of f consists of Hamiltonian stable functions. However, each of these, by Lemma 8, also satisfies the hypotheses of Remark 6, and hence fails to be topologically stable in the sense of [**dPW**]. $\square$

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