DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **4**, Number **2**, April **1998** pp. 301–320

MINIMAL SETS OF PERIODS FOR TORUS MAPS

Boju Jiang

Department of Mathematics
Peking University, Beijing 100871, China

Jaume Llibre

Departament de Matemàtiques
Universitat Autònoma de Barcelona
Bellaterra, 08193 Barcelona, Spain

Abstract. Let T^r be the r-dimensional torus, and let $f : T^r \to T^r$ be a map. If $\mathrm{Per}(f)$ denotes the set of periods of f, the minimal set of periods of f, denoted by $\mathrm{MPer}(f)$, is defined as $\bigcap_{g\simeq f} \mathrm{Per}(g)$ where $g : T^r \to T^r$ is homotopic to f. First, we characterize the set $\mathrm{MPer}(f)$ in terms of the Nielsen numbers of the iterates of f. Second, we distinguish three types of the set $\mathrm{MPer}(f)$ and show that for each type and any given dimension r, the variation of $\mathrm{MPer}(f)$ is uniformly bounded in a suitable sense. Finally, we classify all the sets $\mathrm{MPer}(f)$ for self-maps of the 3-dimensional torus.

1. Introduction. In dynamical systems there is a growing number of results of the form that certain simple topological hypotheses force qualitative and quantitative properties of the system. Perhaps the best known is "Period three implies chaos" for maps of the interval [13]. The present paper deals with the problem of determining the set of periods (of the periodic orbits) of a map given the homotopy class of the map.

To fix terminology, let $f : X \to X$ be a self-map of a compact connected polyhedron X, and n be a natural number. Let $\mathrm{Fix}(f)$ be the fixed point set of f, and $P_n(f)$ the set of periodic points with least period n, i.e.

$$\begin{aligned}\mathrm{Fix}(f) &:= \{x \in X \mid x = f(x)\},\\ P_n(f) &:= \{x \in X \mid x = f^n(x) \text{ but } x \neq f^k(x) \text{ for any } k < n\}\\ &= \mathrm{Fix}(f^n) \setminus \bigcup_{k<n} \mathrm{Fix}(f^k).\end{aligned}$$

Denote by $\mathrm{Per}(f)$ the set of natural numbers corresponding to least periods of periodic orbits of f, i.e.

$$\mathrm{Per}(f) := \{n \in \mathbf{N} \mid P_n(f) \neq \emptyset\}.$$

When a map $g : X \to X$ is homotopic to f, we shall write $g \simeq f : X \to X$. Define the *minimal set of periods of* f to be the set

$$\mathrm{MPer}(f) := \bigcap_{g\simeq f} \mathrm{Per}(g).$$

AMS 1991 Subject Classification: Primary 55M20, Secondary 57N05, 57N10.

Key words and phrases: minimum set of periods, periodic points, torus maps, torus homeomorphisms, three dimensional torus, Nielsen fixed point theory.