LIMIT CYCLES OF PLANAR DISCONTINUOUS PIECEWISE LINEAR HAMILTONIAN SYSTEMS WITHOUT EQUILIBRIA SEPARATED BY NON-REGULAR CURVES

JOHANA JIMENEZ[1], JAUME LLIBRE[2] AND CLAUDIA VALLS[3]

Abstract. The problem of determining the existence, the maximum number and the position of the limit cycles of the planar discontinuous piecewise linear differential systems is an important problem in the qualitative theory of the differential systems. This is due mainly to the fact that these piecewise differential systems have many applications in mechanical systems, electrical circuits, control theory, ... In this paper we study two families of piecewise linear Hamiltonian systems without equilibria in $\mathbb{R}^2$ separated by a non-regular curve. We provide the maximum number of crossing limit cycles that each family can have and show that this maximum is reached. In this way we are solving for each family the extended 16th Hilbert problem.

1. Introduction and statement of the main results

A *discontinuous piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth curve Σ. The *line of discontinuity* Σ of the discontinuous piecewise differential system is given by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a C^1 function having 0 as a regular value. Observe that Σ is the boundary between the regions $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) > 0\}$ and $\Sigma^- = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) < 0\}$. Hence

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0, \end{cases} \tag{1}$$

is the vector field corresponding to a piecewise differential system with line of discontinuity Σ.

When the vector fields X and Y coincide on the line Σ we have a *continuous piecewise differential system* on $\mathbb{R}^2$, that in general it will not be smooth on Σ.

The vector field (1) usually is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, if the separation line Σ is known. In order to establish a definition for the trajectories of Z, we must have a criterion for the transition of the trajectories between Σ^+ and Σ^- across the curve of discontinuity Σ. The contact between the curve of discontinuity Σ and the vector field X (or Y) is described by the directional derivative of h with respect to the vector field X, i.e.

$$Xh(p) = \langle \nabla h(p), X(p) \rangle .$$

Here $\langle .,. \rangle$ denotes the usual inner product of the plane $\mathbb{R}^2$. Filippov in [8] stated the main results of the discontinuous piecewise differential systems. The curve of discontinuity Σ is divided into the three following sets:

(a) $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$, the *Crossing set*.
(b) $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$, the *Escaping set*.
(c) $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$, the *Sliding set*.

2010 *Mathematics Subject Classification.* Primary 34C29, 34C25, 47H11.
Key words and phrases. Crossing limit cycles, discontinuous piecewise linear Hamiltonian systems, non-regular curve.

The points of Σ where both vector fields X and Y simultaneously point outwards or inwards define the *escaping* Σ^e or *sliding* Σ^s regions, while the interior in Σ of their complement defines the *crossing region* Σ^c (see Figure 1). The points of Σ with are not in $\Sigma^c \cup \Sigma^e \cup \Sigma^s$ are the *tangency* points between X or Y with Σ.

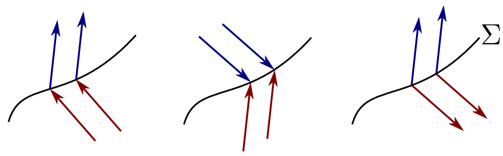


FIGURE 1. Crossing, sliding and escaping regions, respectively.

A *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbits of the system. The limit cycles play a main role in the qualitative theory of the planar differential equations.

It is well known that if the vector fields X and Y are linear they cannot have limit cycles, but the piecewise vector field $Z = (X, Y, \Sigma)$ can have limit cycles. If the limit cycle only contains isolated points of Σ we say that it is a *crossing limit cycle*. In this paper we only will study crossing limit cycles, that frequently we only mention them as limit cycles.

The study of the piecewise linear differential systems is a problem that started with Andronov, Vitt and Khainkin [1] in the 1920's, and nowadays is an important problem in the qualitative theory of the differential systems mainly due to its applications to many physical phenomena, see for instance the books of [7, 16, 17, 23] and the survey [22], and the hundred of papers cited inside these references.

Many of the study developed on the piecewise linear differential systems were done considering piecewise linear differential systems with only two zones and separated by a straight line. Few studies have been done with more zones or considering discontinuity curves different to a straight line.

In 1990 Lum and Chua[20, 21] conjectured that the planar continuous piecewise differential systems separated by a straight line have at most one limit cycle. In 1998 Freire, Ponce, Rodrigo and Torres [10] gave a proof for this conjecture, a shorter proof was given later on by Llibre, Ordóñez and Ponce in [18]. While that for discontinuous piecewise differential systems separated by a straight line Han and Zhang [11] in 2010 conjectured that these systems can exhibit at most two limit cycles. A numerical counterexample to this conjecture was given by Huan and Yang [13] in 2012 providing a discontinuous piecewise differential systems separated by a straight line with three limit cycles. In 2012 Llibre and Ponce [19] proved analytically the existence of these three limit cycles, and later on several authors also find discontinuous piecewise differential systems separated by a straight line with three limit cycles. But we still do not know if three is the maximum number of limit cycles for this class of piecewise differential systems.

The famous and unsolved 16th Hilbert problem asked for an upper bound on the maximum number of limit cycles that polynomial differential systems of a given degree can exhibit, see [12]. These last years several authors have extended this problem to different classes of differential systems in particular to the discontinuous piecewise differential systems formed by distinct linear differential systems. Thus recently in [3, 4, 14, 15] the authors studied the extension of the 16th Hilbert problem for discontinuous piecewise differential systems formed by linear centers which have either two or more zones and they are separated by either conics, or reducible cubics, or irreducible cubics. In [2, 5, 6, 9] the authors considered discontinuous piecewise linear Hamiltonian systems without equilibrium points where such systems are separated by either two parallel straight lines, or conics, or reducible cubics, or irreducible cubics, in each of these classes of piecewise differential systems the authors determined the maximum number of crossing limit cycles that these piecewise linear systems can exhibit.

The results that were obtained in the mentioned papers shown that the shape of the discontinuity curve plays an important role in the number of limit cycles that the piecewise differential systems can have.

The objective of this paper is to study the maximum number of crossing limit cycles for two families of discontinuous piecewise linear Hamiltonian systems without equilibrium points in $\mathbb{R}^2$ separated by a non-regular curve.

First in subsection 1.1 we present our results on the limit cycles of discontinuous piecewise linear Hamiltonian systems without equilibrium points in $\mathbb{R}^2$ where the discontinuity curve is the non-regular line

$$\Sigma_A = \big\{(x, y) \in \mathbb{R}^2 : xy = 0 \text{ and } y \geq 0,\ x \geq 0\big\}.$$

After in subsection 1.2 we present our results on the limit cycles of discontinuous piecewise linear Hamiltonian systems without equilibrium points in $\mathbb{R}^2$ where the discontinuity curve is the non-regular line

$$\Sigma_B = \Big\{(x, y) \in \mathbb{R}^2 : xy = 0 \text{ and } y \geq 0\Big\}.$$

1.1. **Crossing limit cycles intersecting the discontinuity curve Σ_A.** We observed that $\Sigma_A = \Sigma_x^+ \cup \Sigma_y^+$, where

$$\Sigma_x^+ = \big\{(x, y) \in \mathbb{R}^2 : y = 0, x \geq 0\big\} \text{ and } \Sigma_y^+ = \big\{(x, y) \in \mathbb{R}^2 : x = 0, y \geq 0.\big\}$$

The discontinuity curve Σ_A separates the plane $\mathbb{R}^2$ into the following two pieces

$$R_A^1 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\},\ R_A^2 = \mathbb{R}^2 \setminus (R_A^1 \cup \Sigma_A).$$

We denote by $\mathcal{F}_A$ the class of discontinuous piecewise linear differential systems formed by two linear Hamiltonian systems without equilibrium points and separated by Σ_A. We shall study the following class of limit cycles.

A *crossing limit cycle of type A_1* is a limit cycle that intersects Σ_x^+ and Σ_y^+ in exactly one point.

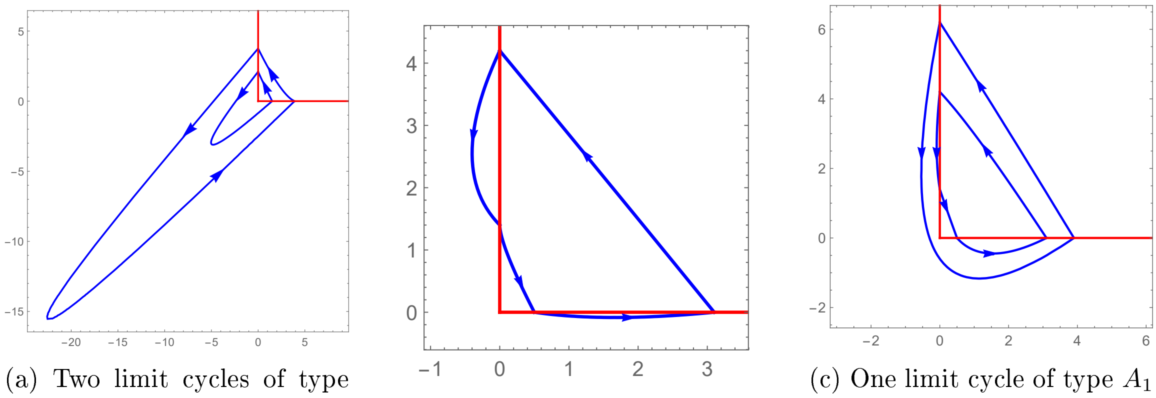
A *crossing limit cycle of type A_2* is a limit cycle that intersects Σ_x^+ and Σ_y^+ in exactly two point.

We observed that it is not possible to have limit cycles of systems in $\mathcal{F}_A$ that intersect only either Σ_x^+, or Σ_y^+, see for instance [9].

Theorem 1. *The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity curve is Σ_A.*

(i) *The maximum number of crossing limit cycles of type A_1 is two. See Figure 2a.*
(ii) *The maximum number of limit cycles of type A_2 is one. See Figure 2b.*
(iii) *There are examples of discontinuous piecewise linear Hamiltonian systems having simultaneously one limit cycle of type A_1 and one limit cycle of type A_2. See Figure 2c.*

Note that although the maximum number of possible simultaneous crossing limit cycles of types A_1 and A_2 are two and one respectively we could not found an example realizing this upper bound. To obtain a limit cycle of a given type it is not enough to obtain the points where the candidate to be a limit cycle intersects the line of discontinuity in the prescribed number, the orbits through these points must close must be well oriented and forming a



(a) Two limit cycles of type A_1.

(b) One limit cycle of type A_2.

(c) One limit cycle of type A_1 and one limit cycle of type A_2.

FIGURE 2. Crossing limit cycles in the family $\mathcal{F}_A$.

periodic orbit, and this periodic orbit must be isolated in order to define a limit cycle. This implies that the search of examples can be very difficult and that the upper bound obtained by our method which does not take into account all these matters of orientation of the orbits, isolateness and pieces of orbits in the prescribed regions could not be optimal.

Theorem 1 is proved in section 3.

1.2. Crossing limit cycles intersecting the discontinuity curve Σ_B.

We observed that $\Sigma_B = \Sigma_x \cup \Sigma_y^+$, where $\Sigma_x = \Sigma_x^+ \cup \Sigma_x^-$, being

$$\Sigma_x^- = \left\{(x,y) \in \mathbb{R}^2 : y = 0, x \leq 0\right\}.$$

The discontinuity curve Σ_B separates the plane $\mathbb{R}^2$ into the following three pieces

$$R_B^1 = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}, \quad R_B^2 = \{(x,y) \in \mathbb{R}^2 : x < 0, y > 0\},$$
$$R_B^3 = \{(x,y) \in \mathbb{R}^2 : y < 0\}.$$

We denote by $\mathcal{F}_B$ the class of discontinuous piecewise linear differential systems formed by three linear Hamiltonian systems without equilibrium points and separated by Σ_B. We shall study the following types of limit cycles.

A *crossing limit cycle of type* B_1 is a limit cycle that intersects Σ_x^+, Σ_x^- and Σ_y^+ in exactly one point.

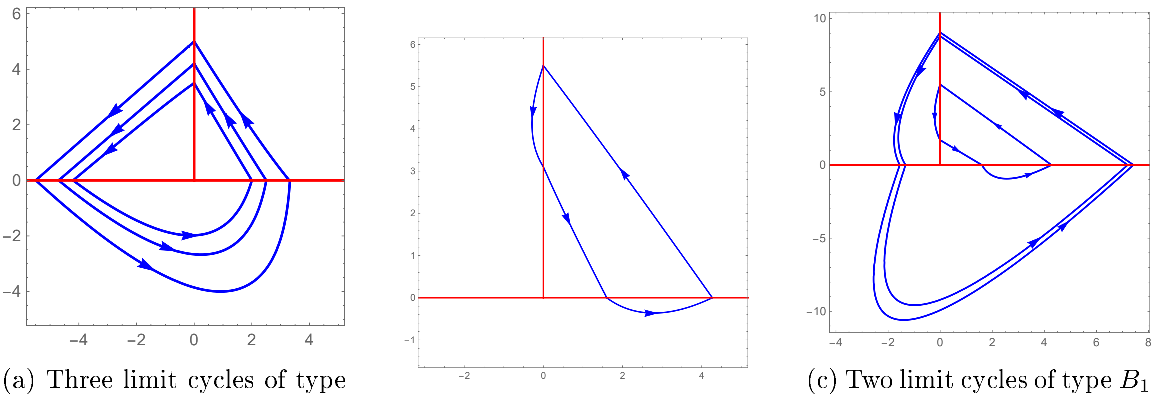
A *crossing limit cycle of type* B_2 is a limit cycle that intersects either Σ_x^+ and Σ_y^+, or Σ_x^- and Σ_y^+ in exactly two points.

Theorem 2. *The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity curve is* Σ_B.

(i) *The maximum number of crossing limit cycles of type* B_1 *is three. See Figure* 3*a.*
(ii) *The maximum number of crossing limit cycles of type* B_2 *is one. See Figure* 3*b.*
(iii) *There are examples of discontinuous piecewise linear Hamiltonian systems having simultaneously two limit cycles of type* B_1 *and one limit cycle of type* B_2*. See Figure* 3*c.*

Note that although the maximum number of possible simultaneous crossing limit cycles of types B_1 and B_2 are three and one respectively we could not found an example realizing this upper bound.

Theorem 2 is proved in section 4.



(a) Three limit cycles of type B_1.

(b) One limit cycle of type B_2.

(c) Two limit cycles of type B_1 and one limit cycle of type B_2.

FIGURE 3. Crossing limit cycles in the family $\mathcal{F}_B$.

2. Preliminary result

In this paper we consider the normal form for an arbitrary linear differential Hamiltonian system in $\mathbb{R}^2$ without equilibrium points provided in [9].

Lemma 1. *The normal form for a linear differential Hamiltonian system in $\mathbb{R}^2$ without equilibria is*

$$\dot{x} = -\lambda bx + by + \gamma, \quad \dot{y} = -\lambda^2 bx + \lambda by + \delta, \quad \textit{with} \quad \delta \neq \lambda\gamma, \quad \textit{and} \quad b \neq 0. \tag{2}$$

This linear differential system has the first integral

$$H(x,y) = -\frac{1}{2}\lambda^2 bx^2 + \lambda bxy - \frac{b}{2}y^2 + \delta x - \gamma y. \tag{3}$$

Note that in the differential system (2) we can assume that $b = 1$ doing a rescaling of the time.

3. Proof of Theorem 1

Proof of statement (i) of Theorem 1. A piecewise linear differential system in the family $\mathcal{F}_A$ is formed by one linear differential Hamiltonian without equilibrium points in each region R_A^1 and R_A^2. Hence by Lemma 1 the general equations for this class of piecewise differential systems are

$$\begin{aligned} \dot{x} &= -\lambda_1 x + y + \gamma_1, \quad \dot{y} = -\lambda_1^2 x + \lambda_1 y + \delta_1, \quad \text{in} \quad R_A^1, \\ \dot{x} &= -\lambda_2 x + y + \gamma_2, \quad \dot{y} = -\lambda_2^2 x + \lambda_2 y + \delta_2, \quad \text{in} \quad R_A^2. \end{aligned}$$

Moreover by Lemma 1 we have that

$$H_i(x,y) = -\frac{1}{2}\lambda_i^2 x^2 + \lambda_i xy - \frac{y^2}{2} + \delta_i x - \gamma_i y, \quad i = 1,2,$$

is a first integral of this system in the region R_A^i for $i = 1,2$.

Now if there is a crossing limit of type A_1, this intersects the discontinuity curve Σ_A in two points $(X,0)$ and $(0,Y)$ with $X > 0$ and $Y > 0$. Of course these two points must satisfy the following closing equations

$$\begin{aligned} e_1 &= H_1(X,0) - H_1(0,Y) = Y^2 + 2Y\gamma_1 + 2X\delta_1 - X^2\lambda_1^2 = 0, \\ e_2 &= H_2(X,0) - H_2(0,Y) = Y^2 + 2Y\gamma_2 + 2X\delta_2 - X^2\lambda_2^2 = 0. \end{aligned} \tag{4}$$

It follows from Bézout theorem that system (4) has at most four real solutions, but since one solution is $(0,0)$, which cannot produce a limit cycle of type A_1, we conclude that system

(4) has at most three real solutions that without loss of generality we can assume that they are $(X_i, 0), (0, Y_i)$ with $i = 1, 2, 3$ satisfying

$$0 < X_1 < X_2 < X_3 \quad \text{and} \quad 0 < Y_1 < Y_2 < Y_3,$$

(otherwise the solutions would intersect which is not possible by the uniqueness of solutions of a differential system).

We consider two different cases.

If $\lambda_1 = \lambda_2 = 0$ then $e_1 = 0$ is a parabola symmetric with respect to some horizontal straight line and passing through the origin. Moreover, $E_2 = e_1 - e_2 = 0$ become

$$E_2 = 2(\delta_1 - \delta_2)X + 2(\gamma_1 - \gamma_2)Y = 0,$$

which is a straight line. Since a parabola and a straight line intersect at most in two points we have that the system $e_1 = E_2 = 0$ intersect at most in two points where one of these two is the origin, then we have at most one point satisfying $0 < X_1$ and $0 < Y_1$, therefore there is at most one limit cycle.

If $\lambda_1\lambda_2 \neq 0$ it follows from $e_1 = e_2 = 0$ that $E_1 = \lambda_2^2 e_1 - \lambda_1^2 e_2 = 0$ and $E_2 = e_1 - e_2 = 0$ become

$$\begin{aligned} E_1 &:= 2(\lambda_2^2\delta_1 - \lambda_1^2\delta_2)X + 2(\lambda_2^2\gamma_1 - \lambda_1^2\gamma_2)Y + (\lambda_2^2 - \lambda_1^2)Y^2 = 0, \\ E_2 &:= 2(\delta_1 - \delta_2)X + 2(\gamma_1 - \gamma_2)Y + (\lambda_2^2 - \lambda_1^2)X^2 = 0. \end{aligned}$$

If $\lambda_2^2 = \lambda_1^2$ (i.e., $\lambda_2 = \pm\lambda_1$) then equations $E_1 = E_2 = 0$ have at most one solution and so there is at most one limit cycle.

Assume now that $\lambda_2 \neq \pm\lambda_1$.

If $\lambda_2^2\delta_1 - \lambda_1^2\delta_2 = 0$ then $E_1 = 0$ reduces to either one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin. The equation $E_2 = 0$ is either a parabola symmetric with respect to some vertical straight line, or one vertical straight line, or two vertical parallel straight lines passing one of these two straight lines through the origin. Since $E_1 = E_2 = 0$ pass through the origin, there are at most two intersection points satisfying $0 < X_1 < X_2$, but $0 < Y_1 = Y_2$ and so there is at most one limit cycle.

If $\gamma_1 - \gamma_2 = 0$ then $E_2 = 0$ reduces to either one vertical straight line, or two vertical parallel straight lines passing one of these two straight lines through the origin. The equation $E_1 = 0$ is either a parabola symmetric with respect to some horizontal straight line, or one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin. Since $E_1 = E_2 = 0$ pass through the origin, there are at most two intersection points satisfying $0 < Y_1 < Y_2$, but $0 < X_1 = X_2$ and so there is at most one limit cycle.

Finally, assume that $\lambda_2^2\delta_1 - \lambda_1^2\delta_2 \neq 0$ and $\gamma_1 - \gamma_2 \neq 0$. In this case, $E_1 = 0$ is a parabola symmetric with respect to some horizontal straight line and $E_2 = 0$ is a parabola symmetric with respect to some vertical line. Since both parabolas intersect at the origin, the maximum number of intersections points is three. When we have three intersection points they satisfy $0 < X_1 < X_2 < X_3$ and $0 < Y_3 < Y_2 < Y_1$, and then there cannot be three limit cycles. Hence there are at most two intersection points satisfying $0 < X_1 < X_2$ and $0 < Y_1 < Y_2$, and so at most two limit cycles.

Now we verify that this found upper bound is reached. We provide a discontinuous piecewise linear differential system in $\mathcal{F}_A$ having exactly two limit cycles of type A_1. In the region R_A^1 we consider the linear Hamiltonian system

$$\dot{x} = \frac{271}{100}x + y - \frac{1837341}{95000}, \quad \dot{y} = -\frac{73441}{10000}x - \frac{271}{100}y + \frac{118235781}{3800000},$$

which has the first integral

$$H_1(x,y) = -\frac{73441}{20000}x^2 - \frac{271}{100}xy + \frac{118235781}{3800000}x - \frac{1}{2}y^2 + \frac{1837341}{95000}y;$$

in region R_A^2 we consider the Hamiltonian system

$$\dot{x} = \frac{1207}{50}x - 34y + \frac{1001997}{47500}, \quad \dot{y} = \frac{85697}{5000}x - \frac{1207}{50}y + \frac{14426523}{1900000}$$

with the first integral

$$H_2(x,y) = -\frac{85697}{10000}x^2 + \frac{1207}{50}xy - \frac{14426523}{1900000}x - 17y^2 + \frac{1001997}{47500}y.$$

With these two systems we obtain that system (4) has exactly two reals solutions (X_j, Y_j), with $X_j > 0$ and $Y_j > 0$, for $j = 1, 2$, namely

$$(X_1, Y_1) = \left(\frac{3}{2}, \frac{21}{10}\right) \quad \text{and} \quad (X_2, Y_2) = \left(\frac{39}{10}, \frac{15}{4}\right),$$

which generated the two crossing limit cycles of type A_1 in Figure 2a. □

Proof of statement (ii) of Theorem 1. If there exists a periodic solution candidate to be a limit cycle of type A_2, then this periodic solution has four intersection points on the discontinuity line Σ_A of the form $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, satisfying $0 < x_1 < x_2$, $0 < y_1 < y_2$. Hence the following equations must be satisfied

$$\begin{aligned} f_1 &= H_1(x_1,0) - H_1(0,y_1) = 2\delta_1 x_1 + 2\gamma_1 y_1 - \lambda_1^2 x_1^2 + y_1^2 = 0,\\ f_2 &= H_1(x_2,0) - H_1(0,y_2) = 2\delta_1 x_2 + 2\gamma_1 y_2 - \lambda_1^2 x_2^2 + y_2^2 = 0,\\ f_3 &= H_2(x_1,0) - H_2(x_2,0) = (x_1 - x_2)(-2\delta_2 + \lambda_2^2 x_1 + \lambda_2^2 x_2) = 0,\\ f_4 &= H_2(0,y_1) - H_2(0,y_2) = (y_2 - y_1)(2\gamma_2 + y_1 + y_2) = 0. \end{aligned} \tag{5}$$

We consider two different cases.

Case 1*:* $\lambda_2 = 0$. In this case from $f_3 = 0$ we get $\delta_2 = 0$. This is not possible, since by Lemma 1 we have that $\delta_2 \neq \lambda_2\gamma_2$.

Case 2*:* $\lambda_2 \neq 0$. From $f_3 = 0$ and $f_4 = 0$, we get

$$x_2 = \frac{2\delta_2}{\lambda_2^2} - x_1, \qquad y_2 = -2\gamma_2 - y_1. \tag{6}$$

Substituting these last expressions in f_1 and f_2 and then setting $F_3 = f_1 - f_2 = 0$, we obtain that system (5) reduces to

$$\begin{aligned} f_2 &= -\gamma_1^2 + \tfrac{\delta_1^2}{\lambda_1^2} - \left(\lambda_1 x_1 - \tfrac{2\delta_2\lambda_1^2 - \delta_1\lambda_2^2}{\lambda_1\lambda_2^2}\right)^2 + ((2\gamma_2 - \gamma_1) + y_1)^2 = 0,\\ F_3 &= \gamma_1\gamma_2 - \gamma_2^2 + \tfrac{\delta_2(\delta_2\lambda_1^2 - \delta_1\lambda_2^2)}{\lambda_2^4} + \left(\delta_1 - \frac{\delta_2\lambda_1^2}{\lambda_2^2}\right)x_1 + (\gamma_1 - \gamma_2)y_1 = 0. \end{aligned} \tag{7}$$

We observed that equation $F_3 = 0$ is a straight line and equation $f_2 = 0$ is a hyperbola.

By Bézout theorem system (7) has at most two real solutions. If they are two real solutions $(x_1^{\pm}, y_1^{\pm})$ from (6) we obtain the correspondent $(x_2^{\pm}, y_2^{\pm})$. Let $(x_1^+, x_2^+, y_1^+, y_2^+)$ be the four intersection points of a first limit cycle of type A_2 with $0 < x_1^+ < x_2^+$, $0 < y_1^+ < y_2^+$, and $(x_1^-, x_2^-, y_1^-, y_2^-)$ be the four intersection points of a second limit cycle of type A_2 with $0 < x_1^- < x_2^-$ and $0 < y_1^- < y_2^-$. If there are two limit cycles of type A_2, we can suppose without loss of generality that the first limit cycle is located in the interior region enclosed by the second limit cycle. Then the intersection points must satisfy

$$0 < x_1^- < x_1^+ < x_2^+ < x_2^- \quad \text{and} \quad 0 < y_1^- < y_1^+ < y_2^+ < y_2^- . \tag{8}$$

We claim that it is not possible to have two solutions $(x_1^{\pm}, y_1^{\pm})$ of system (7) which satisfy the conditions in (8).

If $\gamma_1 - \gamma_2 = 0$ it follows from $F_3 = 0$ that $x_1 = \delta_2/\lambda_2^2$, but then from (6) we get $x_2 = \delta_2/\lambda_2^2$ which is not possible. Hence, $\gamma_1 - \gamma_2 \neq 0$ and solving $F_3 = 0$ in the variable y_1 we obtain

$$y_1 = \frac{-\delta_2^2\lambda_1^2 + \lambda_2^4\left(-\gamma_1\gamma_2 + \gamma_2^2 - \delta_1 x_1\right) + \delta_2\lambda_2^2\left(\delta_1 + \lambda_1^2 x_1\right)}{\lambda_2^4(\gamma_1 - \gamma_2)}.$$

Now we introduce y_1 into $f_2 = 0$ and we get

$$P(x_1) = C_0 + C_1 x_1 + C_2 x_1^2 = 0,$$

being

$$\begin{aligned}
C_0 =& \frac{\left(\gamma_2\lambda_2^4(\gamma_1-\gamma_2) - \delta_1\delta_2\lambda_2^2 + \delta_2^2\lambda_1^2\right)\left(\lambda_2^4\left(-\left(2\gamma_1^2 - 3\gamma_1\gamma_2 + \gamma_2^2\right)\right) - \delta_1\delta_2\lambda_2^2 + \delta_2^2\lambda_1^2\right)}{\lambda_2^8(\gamma_1-\gamma_2)^2}, \\
C_1 =& \frac{2\delta_2\left(\lambda_2^4\left(\lambda_1^2(\gamma_1-\gamma_2)^2 - \delta_1^2\right) + 2\delta_1\delta_2\lambda_1^2\lambda_2^2 - \delta_2^2\lambda_1^4\right)}{\lambda_2^6(\gamma_1-\gamma_2)^2}, \\
C_2 =& \frac{\lambda_2^4\left(\delta_1^2 - \lambda_1^2(\gamma_1-\gamma_2)^2\right) - 2\delta_1\delta_2\lambda_1^2\lambda_2^2 + \delta_2^2\lambda_1^4}{\lambda_2^4(\gamma_1-\gamma_2)^2}.
\end{aligned}$$

The polynomial $P(x_1)$ is quadratic in the variable x_1, whose roots $x_{1,\pm}$, are

$$x_{1,\pm} = \frac{\delta_2}{\lambda_2^2} \pm \frac{\lambda_2^2\sqrt{\Delta/4}}{C_2},$$

where

$$\begin{aligned}
\Delta =& \frac{4}{\lambda_2^8(\gamma_1-\gamma_2)^2}\Bigg(\left(\delta_2\lambda_1^2 - \lambda_2^2\lambda_1(\gamma_1-\gamma_2) + \delta_1)\right)\left(\delta_2\lambda_1^2 - \lambda_2^2(\lambda_1(\gamma_2-\gamma_1) + \delta_1)\right) \\
& \left(\gamma_2\lambda_2^4(2\gamma_1-\gamma_2) - 2\delta_1\delta_2\lambda_2^2 + \delta_2^2\lambda_1^2\right)\Bigg).
\end{aligned}$$

Note that from (6) we get

$$x_{2,\pm} = \frac{2\delta_2}{\lambda_2^2} - x_{1,\pm} = \frac{\delta_2}{\lambda_2^2} \mp \frac{\lambda_2^2\sqrt{\Delta/4}}{C_2} = x_{1,\mp}.$$

The argument of the claim follows. Therefore the maximum number of limit cycles of type A_2 that our discontinuous piecewise linear differential system can have is one. Now we verify that the upper bound found is reached. We provide a discontinuous piecewise linear differential system in $\mathcal{F}_A$ having exactly one limit cycle of type A_2.

In the region R_A^1 we consider the Hamiltonian system

$$\dot{x} = -\frac{3}{2}x - y + \frac{12647}{8960}, \quad \dot{y} = \frac{9}{4}x + \frac{3}{2}y - \frac{327}{128}$$

this system has the first integral

$$H_1(x,y) = -\frac{9}{8}x^2 - \frac{3}{2}xy + \frac{327}{128}x - \frac{1}{2}y^2 + \frac{12647}{8960}y;$$

and in the region R_A^2 we consider the Hamiltonian system

$$\dot{x} = \frac{5}{8}x - y + \frac{14}{5}, \quad \dot{y} = \frac{25}{64}x - \frac{5}{8}y - \frac{45}{64}$$

which has the first integral

$$H_2(x,y) = -\frac{25}{128}x^2 + \frac{5}{8}xy + \frac{45}{64}x - \frac{1}{2}y^2 + \frac{14}{5}y;$$

With these systems in each region R_A^i we have that the system of closing equations (5) has exactly one real solutions (x_1, x_2, y_1, y_2) which satisfy $0 < x_1 < x_2$ and $0 < y_1 < y_2$, namely

$$(x_1, x_2, y_1, y_2) = \left(\frac{1}{2}, \frac{31}{10}, \frac{7}{5}, \frac{22}{5}\right).$$

This real solution generated the crossing limit cycle of type A_2 in the family $\mathcal{F}_A$ in Figure 2b. □

Proof statement (iii) of Theorem 1. Here we provide a piecewise linear Hamiltonian system in $\mathcal{F}_A$ which has exactly one limit cycle of type A_1 and one limit cycle of type A_2.

In region R_A^1 we consider the Hamiltonian system

$$\dot{x} = -\frac{6}{13}\sqrt{\frac{1351}{79}}x - y + \frac{13682}{5135}, \quad \dot{y} = \frac{48636}{13351}x + \frac{6}{13}\sqrt{\frac{1351}{79}}y - \frac{85596}{13351},$$

which has the first integral

$$H_1(x,y) = -\frac{24318}{13351}x^2 - \frac{6}{13}\sqrt{\frac{1351}{79}}xy + \frac{85596}{13351}x - \frac{1}{2}y^2 + \frac{13682}{5135}y;$$

in region R_A^2 we consider the Hamiltonian system

$$\dot{x} = 2\sqrt{\frac{31}{39}}x - y + \frac{14}{5}, \quad \dot{y} = \frac{124}{39}x - 2\sqrt{\frac{31}{39}}y - \frac{372}{65},$$

this Hamiltonian system has the first integral

$$H_2(x,y) = \frac{62}{39}x^2 + 2\sqrt{\frac{31}{39}}xy + \frac{372}{65}x - \frac{1}{2}y^2 + \frac{14}{5}y.$$

With these Hamiltonian systems we have that system (4) has exactly the one real solution

$$(X, Y) = \left(\frac{39}{10}, \frac{33}{5}\right),$$

and system (5) has exactly the one real solution

$$(x_1, x_2, y_1, y_2) = \left(\frac{1}{2}, \frac{31}{10}, \frac{7}{5}, \frac{22}{5}\right).$$

These reals solutions generated one crossing limit cycle of type A_1 and one crossing limit cycle of type A_2 in $\mathcal{F}_A$ in Figure 2c.

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Proof of statement (i) of Theorem 2. We have that a piecewise linear differential system in the family $\mathcal{F}_B$ is formed by one linear differential Hamiltonian without equilibrium points in each region R_B^i for $i = 1, 2, 3$. Hence by Lemma 1 the general equations for this class of piecewise differential systems are

$$\begin{aligned}
&\dot{x} = -\lambda_1 x + y + \gamma_1, \quad \dot{y} = -\lambda_1^2 x + \lambda_1 y + \delta_1, \quad \text{in} \quad R_B^1,\\
&\dot{x} = -\lambda_2 x + y + \gamma_2, \quad \dot{y} = -\lambda_2^2 x + \lambda_2 y + \delta_2, \quad \text{in} \quad R_B^2, \qquad (9)\\
&\dot{x} = -\lambda_3 x + y + \gamma_3, \quad \dot{y} = -\lambda_3^2 x + \lambda_3 y + \delta_3, \quad \text{in} \quad R_B^3,
\end{aligned}$$

and

$$H_i(x,y) = -\frac{1}{2}\lambda_i^2 x^2 + \lambda_i xy - \frac{y^2}{2} + \delta_i x - \gamma_i y, \quad i = 1,2,3$$

is a first integral of the system in the region R_B^i for $i = 1,2,3$.

In order to have a crossing limit of type B_1, it must intersects the discontinuity curve Σ_B in three points $(X,0)$, $(0,Y)$ and $(x,0)$ with $x < 0 < X$ and $Y > 0$. Of course these three points must satisfy the following closing equations

$$(10) \qquad \begin{aligned} \tilde{e}_1 &= H_1(X,0) - H_1(0,Y) = Y^2 + 2Y\gamma_1 + 2X\delta_1 - X^2\lambda_1^2 = 0,\\ \tilde{e}_2 &= H_2(x,0) - H_2(0,Y) = Y^2 + 2Y\gamma_2 + 2x\delta_2 - x^2\lambda_2^2 = 0,\\ \tilde{e}_3 &= H_3(X,0) - H_3(x,0) = (X-x)((x+X)\lambda_3^2 - 2\delta_3) = 0. \end{aligned}$$

Since $X - x \neq 0$ equation $\tilde{e}_3$ reduces to $\tilde{e}_3 = (x+X)\lambda_3^2 - 2\delta_3 = 0$. If $\lambda_3 = 0$, then $\delta_3 = 0$ which is not possible because by Lemma 1 $\delta_3 \neq \lambda_3\gamma_3$. Therefore we have that $\lambda_3 \neq 0$. Isolating x and substituting in equation $\tilde{e}_2$ system (10) reduces to system

$$(11) \qquad \begin{aligned} \tilde{e}_1 &= Y^2 + 2\gamma_1 Y + 2\delta_1 X - \lambda_1^2 X^2 = 0,\\ \tilde{e}_2 &= Y^2 + 2\gamma_2 Y + \frac{2}{\lambda_3^2}\left(2\delta_3\lambda_2^2 - \delta_2\lambda_3^2\right) X - \lambda_2^2 X^2 + 4\frac{\delta_3}{\lambda_3^4}\left(\delta_2\lambda_3^2 - \delta_3\lambda_2^2\right) = 0. \end{aligned}$$

It follows from Bézout theorem that system (11) has at most four real solutions, $(X_i,0), (0,Y_i)$ with $i = 1,2,3,4$ where

$$(12) \qquad 0 < X_1 < X_2 < X_3 < X_4 \quad \text{and} \quad 0 < Y_1 < Y_2 < Y_3 < Y_4.$$

We consider different cases.

If either $\delta_3 = 0$, or $\delta_2\lambda_3^2 - \delta_3\lambda_2^2 = 0$, then equation $\tilde{e}_1 = \tilde{e}_2 = 0$ are hyperbolas passing through the origin. Since both hyperbolas intersect at most in four points being one of these the origin, we have that there is at most three points satisfying condition (12). Therefore there are at most three limit cycles.

We assume that $\delta_3 \neq 0$ and $\delta_2\lambda_3^2 - \delta_3\lambda_2^2 \neq 0$.

If $\lambda_1 = \lambda_2 = 0$, system (9) has at most two limit cycles because the resultant of the polynomial $\tilde{e}_1$ and $\tilde{e}_2$ with respect to the variable X is

$$2Y^2(\delta_1 + \delta_2) + 4Y(\gamma_2\delta_1 + \gamma_1\delta_2) + \frac{8\delta_1\delta_2\delta_3}{\lambda_3^2},$$

which has at most two positive solutions in the variable Y and equation $\tilde{e}_2 = 0$ reduces to

$$Y^2 + 2\gamma_2 Y - 2\delta_2 X + \frac{4\delta_2\delta_3}{\lambda_3\ 2} = 0.$$

Since $\delta_2 \neq 0$, for each solution Y there is at most one solution X of $\tilde{e}_2 = 0$ and so there are at most two limit cycles.

Consider now that $\lambda_1 = 0, \lambda_2 \neq 0$ ($\lambda_1 \neq 0, \lambda_2 = 0$), we have that $\tilde{e}_1 = 0$ reduces to either a parabola symmetric with respect to some horizontal straight line, or one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin (hyperbola passing through the origin). Moreover, $\tilde{e}_2 = 0$ is a hyperbola (is either a parabola symmetric with respect to some horizontal straight line, or one horizontal straight line, or two horizontal parallel straight lines). In both cases we have that there are at most three intersection points satisfying condition (12) and so at most three limit cycles.

Assume now that $\lambda_1\lambda_2 \neq 0$. From $\tilde{e}_1 = 0 = \tilde{e}_2$ it follows that $\tilde{E}_1 = \lambda_2^2\tilde{e}_1 - \lambda_1^2\tilde{e}_2 = 0$ and $\tilde{E}_2 = \tilde{e}_1 - \tilde{e}_2 = 0$ become

$$\begin{aligned}\tilde{E}_1 :=& 4\frac{\delta_3\lambda_1^2(\delta_3\lambda_2^2 - \delta_2\lambda_3^2)}{\lambda_3^4} + 2\left(\lambda_1^2\delta_2 + \lambda_2^2\left(\delta_1 - \frac{2\delta_3\lambda_1^2}{\lambda_3^2}\right)\right)X + 2(\lambda_2^2\gamma_1 - \lambda_1^2\gamma_2)Y \\ &+ (\lambda_2^2 - \lambda_1^2)Y^2 = 0, \\ \tilde{E}_2 :=& 4\frac{\delta_3(\delta_3\lambda_2^2 - \delta_2\lambda_3^2)}{\lambda_3^4} + 2\left(\delta_1 + \delta_2 - \frac{2\delta_3\lambda_2^2}{\lambda_3^2}\right)X + 2(\gamma_1 - \gamma_2)Y + (\lambda_2^2 - \lambda_1^2)X^2 = 0.\end{aligned}$$

If $\lambda_2^2 = \lambda_1^2$ (i.e., $\lambda_2 = \pm\lambda_1$) then equations $\tilde{E}_1 = \tilde{E}_2 = 0$ are straight lines, and so they have at most one solution implying that there is at most one limit cycle.

Assume now that $\lambda_2 \neq \pm\lambda_1$.

If $\gamma_1 - \gamma_2 = 0$ then $\tilde{E}_2 = 0$ reduces to either one vertical straight line, or two vertical parallel straight lines. The equation $\tilde{E}_1 = 0$ is either a parabola symmetric with respect to some horizontal straight line, or one horizontal straight line, or two horizontal parallel straight lines. Then $\tilde{E}_1 = \tilde{E}_2 = 0$, have at most four intersection points but only two intersection points satisfying $0 < X_1 < X_2$ and $0 < Y_1 < Y_2$, and so at most two limit cycles.

If $\lambda_1^2\delta_2 + \lambda_2^2\left(\delta_1 - \dfrac{2\delta_3\lambda_1^2}{\lambda_3^2}\right) = 0$ then $\tilde{E}_1 = 0$ reduces to either one horizontal straight line, or two horizontal parallel straight lines passing. The equation $\tilde{E}_2 = 0$ is either a parabola symmetric with respect to some vertical straight line, or one vertical straight line, or two vertical parallel straight lines. Therefore $\tilde{E}_1 = \tilde{E}_2 = 0$, have at most four intersection points but only two intersection points satisfying $0 < X_1 < X_2$ and $0 < Y_1 < Y_2$, and so at most two limit cycles.

Finally, assume that $\lambda_1^2\delta_2 + \lambda_2^2\left(\delta_1 - \dfrac{2\delta_3\lambda_1^2}{\lambda_3^2}\right) \neq 0$ and $\gamma_1 - \gamma_2 \neq 0$. In this case $\tilde{E}_1 = 0$ is a parabola symmetric with respect to some horizontal straight line and $\tilde{E}_2 = 0$ is a parabola symmetric with respect to some vertical line. Then the maximum number of intersections points is four but we only have three intersection points satisfying $0 < X_1 < X_2 < X_3$ and $0 < Y_1 < Y_2 < Y_3$, and so there are at most three limit cycles.

Now we provide a discontinuous piecewise linear differential system in $\mathcal{F}_B$ which has exactly three crossing limit cycles of type B_1. And this proves that the upper bound found is reached.

In the region R_B^1 we consider the Hamiltonian system

$$\dot{x} = \sqrt{\frac{483}{109}}x + y - \frac{20003}{2180}, \quad \dot{y} = -\frac{483}{109}x - \sqrt{\frac{483}{109}}y + \frac{37989}{2180},$$

this system has the first integral

$$H_1(x,y) = -\frac{483}{218}x^2 - \sqrt{\frac{483}{109}}xy + \frac{37989}{2180}x - y^2 + \frac{20003}{2180}y;$$

in the region R_B^2 we consider the Hamiltonian system

$$\dot{x} = 3\sqrt{\frac{15}{157}}x - y + \frac{10589}{3140}, \quad \dot{y} = \frac{135}{157}x - 3\sqrt{\frac{15}{157}}y + \frac{1983}{628},$$

which has the first integral

$$H_2(x,y) = -\frac{135}{314}x^2 + 3\sqrt{\frac{15}{157}}xy - \frac{1983}{628}x - \frac{1}{2}y^2 + \frac{10589}{3140}y;$$

and finally in region R_B^3 we have the Hamiltonian system

$$\dot{x} = -2x - y + 5, \quad \dot{y} = 4x + 2y + \frac{22}{5},$$

this system has the first integral

$$H_3(x,y) = -2x^2 - 2xy - \frac{22}{5}x - \frac{1}{2}y^2 + 5y.$$

With these system in each region R_B^i we have that the system of closing equations (10) has exactly three real solutions (X_j, Y_j, x^j) such that they satisfy condition (12) and $x^j < 0$, for $j = 1,2,3$, namely

$$(X_1, Y_1, x^1) = \left(2, \frac{7}{2}, -\frac{21}{5}\right), (X_2, Y_2, x^2) = \left(\frac{5}{2}, \frac{21}{5}, -\frac{47}{10}\right), (X_3, Y_3, x^3) = \left(\frac{33}{10}, 5, -\frac{11}{2}\right).$$

These three real solutions generated the three crossing limit cycles of type B_1 in the family $\mathcal{F}_B$ in Figure 3a. $\square$

Proof of statement (*ii*) *of Theorem* 2. If there exists a periodic solution candidate to be a limit cycle of type B_2, then this periodic solution has four intersection points on the discontinuity curve Σ_B of the form $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, satisfying $0 < x_1 < x_2$, $0 < y_1 < y_2$. Hence, the following equations must be satisfied

$$\begin{aligned}
\tilde{f}_1 &= H_1(x_2,0) - H_1(0,y_2) = 2\delta_1 x_2 + 2\gamma_1 y_2 - \lambda_1^2 x_2^2 + y_2^2 = 0,\\
\tilde{f}_2 &= H_1(x_1,0) - H_1(0,y_1) = 2\delta_1 x_1 + 2\gamma_1 y_1 - \lambda_1^2 x_1^2 + y_1^2 = 0,\\
\tilde{f}_3 &= H_2(0,y_2) - H_2(0,y_1) = (y_1 - y_2)(2\gamma_2 + y_1 + y_2) = 0,\\
\tilde{f}_4 &= H_3(x_2,0) - H_3(x_1,0) = (x_1 - x_2)(-2\delta_3 + \lambda_3^2 x_1 + \lambda_3^2 x_2) = 0
\end{aligned} \tag{13}$$

We observe that $\lambda_3 \neq 0$, since if $\lambda_3 = 0$ from $\tilde{f}_4 = 0$ we get that $\delta_3 = 0$ and this is not possible because by Lemma 1 we have that $\delta_3 \neq \lambda_3\gamma_3$.

From $\tilde{f}_3 = 0$ and $\tilde{f}_4 = 0$ we get that

$$x_2 = \frac{2\delta_3 - \lambda_3^2 x_1}{\lambda_3^2}, \qquad y_2 = -y_1 - 2\gamma_2. \tag{14}$$

Using these last expressions we can write the first and second equations of (13) in terms of x_1 and y_1, and we get

$$\begin{aligned}
\tilde{f}_1(x_1,y_1) =& 4\left(\gamma_2^2 - \gamma_1\gamma_2 + \frac{\delta_3\left(\delta_1\lambda_3^2 - \delta_3\lambda_1^2\right)}{\lambda_3^4}\right) + \left(\frac{4\delta_3\lambda_1^2}{\lambda_3^2} - 2\delta_1\right)x_1 + 2(2\gamma_2 - \gamma_1)y_1\\
&- \lambda_1^2 x_1^2 + y_1^2 = 0,\\
\tilde{f}_2(x_1,y_1) =& 2\delta_1 x_1 + 2\gamma_1 y_1 - \lambda_1^2 x_1^2 + y_1^2 = 0.
\end{aligned}$$

Setting $\tilde{F}_3(x_1,y_1) = \tilde{f}_2(x_1,y_1) - \tilde{f}_1(x_1,y_1) = 0$ we obtain

$$\tilde{F}_3(x_1,y_1) = 4\left(\left(\lambda_3^2\gamma_1\gamma_2 - \lambda_3^2\gamma_2^2 - \delta_1\delta_3 + \frac{\lambda_1^2\delta_3^2}{\lambda_3^2}\right) - (\lambda_1^2\delta_3 - \lambda_3^2\delta_1)x_1 + \lambda_3^2(\gamma_1 - \gamma_2)y_1\right) = 0.$$

If $\gamma_1 - \gamma_2 = 0$ it follows from $\tilde{F}_3(x_1,y_1) = 0$ that $x_1 = \delta_3/\lambda_3^2$, but then from (14) we get $x_2 = \delta_3/\lambda_3^2 = x_1$ which is not possible. Hence $\gamma_1 - \gamma_2 \neq 0$ and solving $\tilde{F}_3(x_1,y_1) = 0$ in the variable y_1 we obtain

$$y_1 = \frac{\gamma_2\lambda_3^4(\gamma_2 - \gamma_1) + \delta_1\delta_3\lambda_3^2 - \delta_3^2\lambda_1^2 + \lambda_3^2(\delta_3\lambda_1^2 - \delta_1\lambda_3^2)x_1}{(\gamma_1 - \gamma_2)\lambda_3^4}.$$

Now we substituting y_1 into $\tilde{f}_2(x_1,y_1) = 0$ we have

$$P(x_1) = \tilde{C}_0 + \tilde{C}_1 x_1 + \tilde{C}_2 x_1^2 = 0,$$

where

$$\tilde{C}_0 = \frac{\left(\gamma_2\lambda_3^4(\gamma_1-\gamma_2)-\delta_1\delta_3\lambda_3^2+\delta_3^2\lambda_1^2\right)\left(-\lambda_3^4\left(2\gamma_1^2-3\gamma_1\gamma_2+\gamma_2^2\right)-\delta_1\delta_3\lambda_3^2+\delta_3^2\lambda_1^2\right)}{\lambda_3^8(\gamma_1-\gamma_2)^2},$$

$$\tilde{C}_1 = \frac{2\delta_3\left(\lambda_3^4\left(\lambda_1^2(\gamma_1-\gamma_2)^2-\delta_1^2\right)+2\delta_1\delta_3\lambda_1^2\lambda_3^2-\delta_3^2\lambda_1^4\right)}{\lambda_3^6(\gamma_1-\gamma_2)^2},$$

$$\tilde{C}_2 = \frac{\lambda_3^4\left(\delta_1^2-\lambda_1^2(\gamma_1-\gamma_2)^2\right)-2\delta_1\delta_3\lambda_1^2\lambda_3^2+\delta_3^2\lambda_1^4}{\lambda_3^4(\gamma_1-\gamma_2)^2}.$$

The polynomial $P(x_1)$ is quadratic in the variable x_1, whose roots $x_1^{\pm}$, are

$$x_1^{\pm} = \frac{2\delta_3\left(-\tilde{C}_2\delta_3 \pm \sqrt{\frac{\Delta}{4}}\lambda_3^2\right)}{\tilde{C}_1\lambda_3^4},$$

where

$$\Delta = \frac{4}{\lambda_3^8(\gamma_1-\gamma_2)^2}\left(\delta_3\lambda_1^2-\lambda_3^2(\lambda_1(\gamma_1-\gamma_2)+\delta_1)\right)\left(\delta_3\lambda_1^2-\lambda_3^2(\lambda_1(\gamma_2-\gamma_1)+\delta_1)\right)$$
$$\left(\gamma_2\lambda_3^4(2\gamma_1-\gamma_2)-2\delta_1\delta_3\lambda_3^2+\delta_3^2\lambda_1^2\right).$$

Note that from (14) we get

$$x_2^{\pm} = \frac{2\delta_3-\lambda_3^2x_1^{\pm}}{\lambda_3^2} = \frac{2\delta_3\left(\delta_3\tilde{C}_2+\lambda_3^2\tilde{C}_1 \mp \lambda_3^2\sqrt{\frac{\Delta}{4}}\right)}{\tilde{C}_1\lambda_3^4} = x_1^{\mp}.$$

Since $x_2 > x_1$, there is at most one solution of the two possible solutions $x_1^{\pm}$. Therefore, the maximum number of limit cycles of type B_2 that our discontinuous piecewise linear differential system can have is one.

Now we verify that this upper bound found is reached. We provide a discontinuous piecewise linear differential system in $\mathcal{F}_B$ having exactly one crossing limit cycle of type B_2. And this proves that the upper bound found for this case is reached.

We consider the linear Hamiltonian system

$$\dot{x} = -\frac{3}{2}x - y + \frac{561299}{147900}, \quad \dot{y} = \frac{9}{4}x + \frac{3}{2}y - \frac{7276597}{1183200}$$

in region R_B^1, and this system has the first integral

$$H_1(x,y) = -\frac{9}{8}x^2 - \frac{3}{2}xy + \frac{7276597}{1183200}x - \frac{1}{2}y^2 + \frac{561299}{147900}y;$$

in the region R_B^2 we consider the Hamiltonian system

$$\dot{x} = \frac{5}{8}x - y + \frac{43}{10}, \quad \dot{y} = \frac{25}{64}x - \frac{5}{8}y + \frac{3}{4}$$

which has the first integral

$$H_2(x,y) = -\frac{25}{128}x^2 + \frac{5}{8}xy - \frac{3}{4}x - \frac{1}{2}y^2 + \frac{43}{10}y;$$

and in the region R_B^3 we consider the Hamiltonian system

$$\dot{x} = \frac{22}{9}x - y + 5, \quad \dot{y} = \frac{484}{81}x - \frac{22}{9}y - \frac{71027}{4050}$$

this system has the first integral

$$H_3(x,y) = -\frac{242}{81}x^2 + \frac{22}{9}xy + \frac{71027}{4050}x - \frac{1}{2}y^2 + 5y.$$

With these three systems we have that system (13) has exactly one real solution, namely

$$(x_1, y_1, y_2, x_2) = \left(\frac{8}{5}, \frac{31}{10}, \frac{427}{100}, \frac{11}{2}\right),$$

which generated the crossing limit cycle of type B_2 in Figure 3b. $\square$

Proof statement (iii) of Theorem 2. We provide a piecewise linear Hamiltonian system in $\mathcal{F}_B$ which has exactly two limit cycles of type B_1 and one limit cycle of type B_2.

In region R_B^1 we consider the Hamiltonian system

$$\dot{x} = -\sqrt{\frac{314270}{223671}}x - y + \frac{720859}{213020}, \quad \dot{y} = \frac{314270}{223671}x + \sqrt{\frac{314270}{223671}}y - \frac{17072429}{4473420},$$

which has the first integral

$$H_1(x,y) = -\frac{157135}{223671}x^2 - \sqrt{\frac{314270}{223671}}xy + \frac{17072429}{4473420}x - \frac{1}{2}y^2 + \frac{720859}{213020}y;$$

in region R_B^2 we consider the Hamiltonian system

$$\begin{aligned}
\dot{x} =& \frac{\sqrt{\frac{5\left(782031760005 - 556339\sqrt{1459934861005}\right)}{4123}}}{10651}x - y + \frac{18}{5},\\
\dot{y} =& \frac{5\left(782031760005 - 556339\sqrt{1459934861005}\right)}{467728791523}x - \frac{\sqrt{\frac{5\left(782031760005 - 556339\sqrt{1459934861005}\right)}{4123}}}{10651}y\\
&- \left(\frac{556339\sqrt{1459934861005} - 782031760005}{140670313240} + \frac{704}{133}\right),
\end{aligned}$$

this Hamiltonian system has the first integral

$$\begin{aligned}
H_2(x,y) =& \frac{5\left(782031760005 - 556339\sqrt{1459934861005}\right)}{935457583046}x^2 + \frac{1}{2}y^2 - \frac{18}{5}y\\
&- \frac{\sqrt{\frac{5\left(782031760005 - 556339\sqrt{1459934861005}\right)}{4123}}}{10651}xy\\
&+ \left(\frac{556339\sqrt{1459934861005} - 782031760005}{140670313240} + \frac{704}{133}\right)x;
\end{aligned}$$

and in the region R_B^3 we consider the Hamiltonian system

$$\dot{x} = \frac{22}{9}x - y - 1, \quad \dot{y} = \frac{484}{81}x - \frac{22}{9}y - \frac{71027}{4050},$$

which has the first integral

$$H_3(x,y) = -\frac{242}{81}x^2 + \frac{22}{9}xy + \frac{71027}{4050}x - \frac{1}{2}y^2 - y.$$

With these Hamiltonian systems we have that system (10) has exactly the two real solutions

$$\begin{aligned}
(X_1, Y_1, x^1) &= \left(\frac{36}{5}, \frac{44}{5}, -\frac{133}{100}\right),\\
(X_2, Y_2, x^2) &= \left(\frac{371}{50}, \frac{\sqrt{1459934861005} + 720859}{213020}, -\frac{31}{20}\right),
\end{aligned}$$

and system (13) has exactly the one real solution

$$(x_1, y_1, x_2, y_2) = \left(\frac{427}{100}, \frac{11}{2}, \frac{8}{5}, \frac{17}{10}\right).$$

These reals solutions generated the two crossing limit cycles of type B_1 and one crossing limit cycle of type B_2 in $\mathcal{F}_B$ in Figure 3c.

This completes the proof of Theorem 2. □

Acknowledgements

The second author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is partially supported by FCT/Portugal through UID/MAT /04459/2019.

References

[1] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.
[2] A. Belfar, R. Benterki and J. Llibre, *Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics*, submitted, 2021.
[3] R. Benterki and J. Llibre, *The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **28** (2021), no. 3, 153–192.
[4] R. Benterki and J. Llibre, *The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II*, to appear in Differential Equations and Dynamical Systems, (2021).
[5] R. Benterki and J. LLibre, *Crossing limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points*, Mathematics **2020**(8), 755, 14 pp.
[6] R. Benterki, J.J. Jimenez and J. LLibre, *Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibria separated by reducible cubics*, to appear in Electron. J. Qual. Theory Differ. Equ. (2021)
[7] M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Appl. Math. Sci., vol. 163, Springer-Verlag, London, 2008.
[8] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht, 1988.
[9] A.F. Fonseca, J. Llibre and L.F. Mello, *Limit cycles in planar piecewise linear Hamiltonian systems with three zones without equilibrium points*, Int. J. Bifurcation and Chaos **30**(11), (2020) 2050157, 8 pp.
[10] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.
[11] M. Han and W. Zhang, *On hopf bifurcation in non-smooth planar systems*, J. Differential Equations **248**(9) (2010), 2399–2416.
[12] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ottingen Math. Phys. KL.* (1900), 253–297; English transl., Bull. Amer. Math. Soc. **8** (1902), 437–479; Bull. (New Series) Amer. Math. Soc. **37** (2000), 407–436.
[13] S. M. Huan and X. S. Yang, *On the number of limit cycles in general planar piecewise systems*, Discrete Cont. Dyn. Syst., Series A **32** (2012), 2147–2164.
[14] J.J. Jimenez, J. Llibre and J.C. Medrado, *Crossing limit cycles for a class of piecewise linear differential centers separated by a conic*, Electron. J. Differential Equations **41** (2020), 36 pp.
[15] J.J. Jimenez, J. Llibre and J.C. Medrado, *Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve*, Electron. J. Qual. Theory Differ. Equ.**19** (2020), 48 pp.
[16] R.I. Leine and H. Nijmeijer, *Dynamics and bifurcations of non-smooth mechanical systems*, Lecture Notes in Applied and Computational Mechanics, **18**, Springer-Verlag, Berlin, 2004.
[17] D. Liberzon, *Switching in systems and control: Foundations and Applications*, Birkhäuser, Boston, 2003.
[18] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry*, Nonlinear Analysis: Real World Applications **19** (2013), 325–335.

[19] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dyn. Contin. Discr. Impul. Syst., Ser. B **19** (2012), 325–335.
[20] R. Lum and L.O. Chua, *Global propierties of continuous piecewise-linear vector fields. Part I: Simplest case in* $\mathbb{R}^2$, Int. J. of Circuit Theory and Appl. **19**(3) (1991), 251–307.
[21] R. Lum and L.O. Chua, *Global properties of continuous piecewise linear vector fields. II. Simplest symmetric case in* $\mathbb{R}^2$, Int. J. of Circuit Theory and Appl.**20**(1) (1992), 9–46.
[22] O. Makarenkov and J. S. W. Lamb, *Dynamics and bifurcations of nonsmooth systems: a survey*, Phys. D **241** (2012), 1826–1844.
[23] D.J.W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, World Sci. Ser. Nonlinear Sci. Ser. A, vol. **69**, World Scientific, Singapore, 2010.

[1] Universidade Federal do Oeste da Bahia, 46470000 Bom Jesus da Lapa, Bahia, Brazil

Email address: `jjohanajimenez@gmail.com`

[2] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: `cvalls@math.ist.utl.pt`