Universitat Autònoma de Barcelona

Set of periods, topological entropy and combinatorial dynamics for tree and graph maps

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Per l'Esteve,
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Introduction

This memoir deals with one-dimensional discrete dynamical systems, from both a topological and a combinatorial point of view. More precisely, we are interested in the periodic orbits and topological entropy of continuous self-maps defined on trees and graphs.

The central problem of our work is the characterization of the possible set of periods of all periodic orbits exhibited by a *tree map* (any continuous map from a tree into itself). The widely known Sharkovskii's Theorem (1964) concerning interval maps was the first remarkable result in this setting. This beautiful theorem states that the set of periods of any interval map is an initial segment of the following linear ordering $\trianglerighteq$ in the set $\mathbb{N} \cup \{2^\infty\}$ (the so-called *Sharkovskii ordering*):

$$3 \trianglerighteq 5 \trianglerighteq 7 \trianglerighteq \dots \trianglerighteq 2\cdot 3 \trianglerighteq 2\cdot 5 \trianglerighteq 2\cdot 7 \trianglerighteq \dots \trianglerighteq 4\cdot 3 \trianglerighteq 4\cdot 5 \trianglerighteq 4\cdot 7 \trianglerighteq \dots \trianglerighteq \dots \trianglerighteq 2^n\cdot 3 \trianglerighteq 2^n\cdot 5 \trianglerighteq 2^n\cdot 7 \trianglerighteq \dots \trianglerighteq 2^\infty \trianglerighteq \dots \trianglerighteq 2^n \trianglerighteq \dots \trianglerighteq 16 \trianglerighteq 8 \trianglerighteq 4 \trianglerighteq 2 \trianglerighteq 1.$$

Conversely, given any initial segment $\mathcal{I}$ of the ordering $\trianglerighteq$ there exists an interval map whose set of periods coincides with $\mathcal{I}$.

During the last three decades there have been several attempts to find results similar to that of Sharkovskii for one-dimensional spaces other than the interval (the 3-star and the circle, among them). More recently, the case of maps defined on more general trees has been specially treated. Baldwin's Theorem (1991), which solves the problem in the case of n-stars for any $n \geq 1$, has been one of the most significant advances in this direction. This result states that the set of periods of any n-star map is a finite union of initial segments of n-many partial orderings (*Baldwin orderings*). Conversely, given such a union $\mathcal{I}$ there exists an n-star map whose set of periods is $\mathcal{I}$.

A more detailed chronology of related works, as well as citations to other partial results on this matter, can be found in the Introductions to Chapters 1 and 2.

The main purpose of our research is to describe the generic structure of the set of periods of any tree map $g\colon S \longrightarrow S$ in terms of the combinatorial and topological properties of the tree S: amount and arrangement of endpoints, vertices and edges. In Chapter 1 we make a detailed discussion about which is the more natural approach to this problem, and we propose a strategy consisting on three consecutive stages which can be summarized as follows:

1. For each periodic orbit P of g, calculate the set Λ_P of periods of the corresponding *canonical* (or P-minimal) model $f_P : T_P \longrightarrow T_P$.
2. Prove that Λ_P is contained in the set of periods of each tree map exhibiting an orbit with the *pattern* of P. In particular, $\Lambda_P \subset \text{Per}(g)$.
3. Consider each orbit P of g and its associated Λ_P, and then obtain (by purely number-theoretical arguments) a finite structure of the set of periods of g by arranging adequately the (perhaps uncountable) union of all sets Λ_P.

Observe that this approach depends strongly on the notions of *pattern* (of a finite invariant set) and *minimal model* associated to it. These notions were developed in the context of interval maps and widely used in a number of papers during the last two decades. However, equivalent operative definitions for tree maps were not available until 1997, when Alsedà, Guaschi, Los, Mañosas and Mumbrú proposed to define the pattern of a finite invariant set P essentially as a homotopy class of maps relative to the points of P, and proved (constructively) that there always exists a P-minimal model $f_P\colon T_P \longrightarrow T_P$, that is, a representative of the class displaying several dynamic minimality properties. It is important to remark that the trees S and T_P are not necessarily homeomorphic. This complicates considerably the implementation of the second stage of the above programme, since the only features which are preserved when one compares the maps $g\colon S \longrightarrow S$ and $f_P\colon T_P \longrightarrow T_P$ are the relative positions of the points of P and the way g and f_P act on these points.

In Chapter 1 we carry out the first stage of the above programme. That is, given a periodic orbit P and a P-minimal tree map $f\colon T \longrightarrow T$, we calculate (as large as possible) subsets of the set of periods of f. This task, which has been done by studying the loops of the Markov P-graph of f, is relatively simple when P does not exhibit a certain rotational (or *twist*) behavior around a fixed point of f. When P is twist, we perform a reduction process consisting of what we have called a *sequence of partial reductions* leading up to a periodic orbit P' and a P'-minimal tree map $f'\colon T' \longrightarrow T'$ such that $T' \subset T$, $|P| = k|P'|$ for some $k > 1$, the set of periods of f is essentially the set of periods of f' multiplied by k, and P' is non-twist. By means of this strategy we prove Theorem A, which states that the set of periods of f is, up to an explicitly bounded finite set, the initial segment of a Baldwin ordering starting at $|P|$. We also prove a converse result (Theorem B) which states that, given any set $\mathcal{I}$ of that form, there exists a piecewise monotone tree map whose set of periods coincides with $\mathcal{I}$.

The goal of Chapter 2 is to implement in full the above programme by completing stages 2 and 3. In June 2001 we submitted the work of Chapter 1 to be considered for publication as a paper in International Journal of Bifurcation and Chaos ([5]). Later on, while writing a part of Chapter 2 of this memoir, we realized that using a new simple and powerful argument would allow us to shorten considerably the proofs and improve the obtained results. In particular, with this new approach all the lengthy technical work associated to the construction of a *sequence of partial reductions* is unnecessary. This gave rise to a revised version of the above strategy (with a slightly modified stage 1) which we perform completely in Chapter 2. Despite this new approach overcomes a part of the material of Chapter 1, we have chosen to leave intact the published work.

The main result of Chapter 2 is Theorem C, which tells us that for each tree map $g\colon S \longrightarrow S$ there exists a finite set of sequences $\underline{\mathsf{s}} = (p_1, p_2, \ldots, p_m)$ of positive integers such that the set of periods of g is (up to an explicitly bounded finite set) a finite union of sets of the form

$$\{p_1, p_1p_2, \ldots, p_1p_2\cdots p_{m-1}\} \cup (\mathcal{I}_{\underline{\mathsf{s}}} \setminus p_1p_2\cdots p_m\{2, 3, \ldots, \lambda_{\underline{\mathsf{s}}}\}),$$

where $\lambda_{\underline{\mathsf{s}}}$ is a nonnegative integer and $\mathcal{I}_{\underline{\mathsf{s}}}$ is an initial segment of the Baldwin ordering ${}_{p_1p_2\cdots p_m}\!\geq$. The finite set of sequences which characterizes the set of periods of g depends entirely on the combinatorial properties of the tree S. We also prove a converse result (Theorem D) which asserts that given any finite union $\mathcal{I}$ of sets of the above form there exists a tree map whose set of periods is $\mathcal{I}$.

In Chapter 3 we report some computer experiments on the minimality of the dynamics of canonical models. Chronologically, this work is contemporaneous to Chapter 1. While researching about the set of periods of canonical models, we constructed some computer software to explore how the dynamic minimality translates into some forcing properties of patterns and periods. In a spirit of modular programming, we designed lots of self-contained functions which can be used to implement a wide variety of several-purpose software. Among other, we have functions that:

1. Compute the canonical model of a pattern provided by the user.
2. Calculate the Markov transition matrix associated to a piecewise monotone tree map.
3. Extract all the simple loops of a given length from a Markov transition matrix.
4. Calculate the pattern of a periodic orbit associated to a Markov loop.

The efficient programming of a part of this machinery needs an important theoretical background. In Chapter 3 we list and explain the source code (written in language C) of the most important functions. When required, we also state and prove some results which have been used either to construct the algorithms or to optimize the execution time. The code of other minor routines, which are not interesting from a mathematical point of view, has been listed in the Appendix.

Finally, in Chapter 4 we generalize some results of Block & Coven, Misiurewicz & Nitecki and Takahashi, where the topological entropy of an interval map was approximated by the entropies of its periodic orbits (the entropy of a periodic orbit P, denoted by $h(P)$, is the entropy of a P-minimal model). In Theorem E we show that if $f\colon G \longrightarrow G$ is a graph map then the entropy of f equals $\sup\{h(P) : P$ periodic orbit of f and $|P| > m\}$, for each non-negative integer m. This chapter has been published as a paper in Proceedings of the American Mathematical Society ([4]).

Agraïments.

En aquest món que els mitjans de comunicació qualifiquen (amb grans dosis d'humor negre o de mala fe) de globalitzat i multicultural, és un autèntic luxe poder escriure en català l'únic fragment d'aquesta tesi que serà llegit per tothom.

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