PERIODIC ORBITS FOR A GENERALIZED FRIEDMANN-ROBERTSON-WALKER HAMILTONIAN SYSTEM IN DIMENSION 6

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Abstract. A generalized Friedmann-Robertson-Walker Hamiltonian system is studied in dimension 6. The averaging theory is the tool used to provide sufficient conditions on the six parameters of the system which guarantee the existence of continuous families of period orbits parameterized by the energy.

1. Introduction

In astrophysics the study of the dynamic of galaxies progressed considerably thanks to the discovery of important theories coming from mathematical models, see for instance the articles [2], [7], [10], [13].

Calzeta and Hasi (1993) studied the simplified Friedmann-Robertson-Walker Hamiltonian in dimension 4 given by the Hamiltonian $H = (p_y^2 - p_x^2 + y^2 - x^2)/2 + (bx^2y^2)/2$ which "modeled a universe". They proved analytically and numerically the existence of chaotic motion of the Hamiltonian system associated to the above Hamiltonian. Although this model is too simplified to be considered realistically, it is an interesting testing ground for the implications of chaos in cosmology, see for more details [3]. Hawking [4] and Page [8] considered similar models for understanding the relation between the thermodynamic and cosmological arrow of time, in the area of quantum cosmology.

We study the following generalized classical Friedmann-Robertson-Walker Hamiltonian system in dimension 6.

$$(1) \qquad \begin{aligned} H = \ & \frac{1}{2}(p_y^2 + p_z^2 - p_x^2 + y^2 + z^2 - x^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 + 2cx^2z^2 + dy^4 \\ & +2ey^2z^2 + fz^4), \end{aligned}$$

Note that this Hamiltonian depends on six parameters a,b,c,d,e and f.

When $z = p_z = 0$ the previous Hamiltonian contains the planar classical Friedmann-Robertson-Walker Hamiltonian system studied by Calzeta and Hasi. Its periodic solutions were studied in [6].

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Our goal in this article is to study the periodic solutions in the different energy levels $H = h$ of the Hamiltonian system

$$
\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = -p_x,\\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y,\\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z,\\
\dot{p}_x &= -\frac{\partial H}{\partial x} = x - x(ax^2 + by^2 + cz^2),\\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2),\\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2),
\end{aligned}
\tag{2}
$$

associated to the Hamiltonian (1). The dot denotes derivative with respect to the independent variable t, the time.

We study the existence of periodic orbits of system (2) and we compute them by using the *averaging theory*, see for more details about this tool section 2. Specifically we will provide through the averaging method sufficient conditions on the six parameters a,b,c,d,e and f for ensuring the existence of periodic orbits of Friedmann-Robertson-Walker system (2).

The study of the periodic orbits is of special interest because they are the most simple non–trivial solutions of the system after the equilibrium points, and their stability determines the kind of motion in their neighborhood. Note that the periodic orbits studied in this paper are isolated in every energy level.

The averaging method provides periodic orbits of a perturbed periodic non–autonomous differential system depending on a small parameter ε. Roughly speaking, the problem of finding periodic solutions of a differential system is reduced to find zeros of some convenient finite dimensional function. We check the conditions under which the averaging theory guarantees the existence of periodic orbits, and we find them as a function of the energy. In that manner we can find analytically periodic orbits in any energy level as function of the six parameters of the Friedmann-Robertson-Walker systems (2). The conclusion of our main results are the following theorem and lemma.

Theorem 1. *At every fixed energy level $H = \varepsilon h$ with $h \in \mathbb{R}$ the Friedmann-Robertson-Walker Hamiltonian system (2) has at least*

(A) *one periodic orbit if the following conditions hold*

$$h < 0,\ bc \neq 0,\ (a+b)(3a+b)(a+c)(3a+c) \neq 0,\ \left|\frac{3a+2b}{b}\right| < 1,\ \left|\frac{3a+2c}{c}\right| < 1;$$

(B) *two periodic orbits if one of the following six conditions hold*

(1) $h > 0,\ eb \neq 0,\ (b+3d)(b+d) \neq 0,\ \left|\frac{3d+2b}{b}\right| < 1,\ \left|\frac{2e-3d}{e}\right| < 1;$

(2) $h > 0,\ ec \neq 0,\ (c+3f)(c+f) \neq 0,\ \left|\dfrac{2e-3f}{e}\right| < 1,\ \left|\dfrac{3f+2c}{c}\right| < 1;$

(3) $a+2c+f \neq 0,\ bc-ae-ce+bf \neq 0,\ h(a+2c+f)(c+f) < 0,\ h(a+2c+f)(a+c) > 0,\ (bc-c^2-ae-ce+af+bf)(bc-3c^2-ae-ce+3af+bf) \neq 0,\ \left|\dfrac{3c^2-2bc+2ae+2ce-3af-2bf}{bc-ae-ce+bf}\right| < 1;$

(4) $3a+2c+3f \neq 0,\ (bc+3ae+ce+3bf) \neq 0,\ h(3a+2c+3f)(c+3f) < 0,\ h(3a+2c+3f)(3a+c) > 0,\ c(c^2-3bc+3ae+ce-9af-9bf)(c^2-bc+9ae+3ce-9af-3bf) \neq 0,\ \left|\dfrac{-2bc+c^2+6ae+2ce-9af-6bf}{bc+3ae+ce+3bf}\right| < 1;$

(5) $a+2b+d \neq 0,\ b+d \neq 0,\ bc+cd-ae-be \neq 0,\ h(b+d)(a+2b+d) < 0,\ h(a+b)(a+2b+d) > 0,\ b(3b^2-bc-3ad-cd+ae+be)(b^2-bc-ad-cd+ae+be) \neq 0,\ \left|\dfrac{3b^2-2bc-3ad-2cd+2ae+2be}{bc+cd-ae-be}\right| < 1;$

(6) $3a+2b+3d \neq 0,\ 3cd+3ae+bc+be \neq 0,\ h(b+3d)(3a+2b+3d) < 0,\ h(3a+b)(3a+2b+3d) > 0,\ b(b^2-3bc-9ad-9cd+3ae+be)(b^2-bc-9ad-3cd+9ae+3be) \neq 0\ \left|\dfrac{b^2-2bc-9ad-6cd+6ae+2be}{3cd+3ae+bc+be}\right| < 1.$

(C) *four periodic orbits if one of the following twenty eight conditions hold*

(1) $cd-be-ce+bf \neq 0,\ \ d-2e+f \neq 0,\ \ h(d-2e+f)(f-e) > 0, h(d-2e+f)(d-e) > 0, e(-be+bf+cd-ce+3df-3e^2)(-be+bf+cd-ce+df-e^2) \neq 0,\ \left|\dfrac{-2cd+2be+2ce+3e^2-2bf-3df}{cd-be-ce+bf}\right| < 1;$

(2) $bd-cd-be+bf \neq 0, \left|\dfrac{3d+2b}{b}\right| < 1, h(bd-cd-be+bf)(cd-bf) > 0, hb(bd-cd-be+bf)(d-e) > 0,\ \ be(b+d)(b+3d)(d-2e+f)(ce-bf) \neq 0;$

(3) $3cd+be-ce-3bf \neq 0,\ 3d-2e+3f \neq 0,\ h(3f-e)(3d-2e+3f) > 0,\ h(3d-e)(3d-2e+3f) > 0,\ (-9cd+be+3ce+e^2-3bf-9df)(3cd-3be-ce-e^2+9bf+9df) \neq 0,\ \left|\dfrac{2be+2ce+e^2-6cd-6bf-9df}{3cd+be-ce-3bf}\right| < 1;$

(4) $4bc+3bd+3cd-be+3bf \neq 0,\ hb(4bc+3bd+3cd-be+3bf)(3d-e) > 0,\ h(4bc+3bd+3cd-be+3bf)(4bc+3cd+3bf) > 0,\ eb(b+d)(b+3d)(3d-2e+3f)(4bc+ce+3bf) \neq 0,\ \left|\dfrac{3d+2b}{b}\right| < 1;$

(5) $a+c+e = 0,\ e-c-f \neq 0,\ eh(e-c-f) > 0,\ h(cd+df-ce-e^2)(a+d)(e-c-f) > 0,\ ce\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big) \neq 0,\ h(ce+e^2+ac+af)(a+d)(e-c-f) < 0;$

(6) $a+c+e \neq 0,\ c-e+f = 0,\ c-d+e \neq 0,\ eh(a+c+e) < 0,\ hc(c-d+e) > 0,\ h(ae-ad-cd)(a+c+e)(c-d+e) > 0,\ ce(c^2-ad-2cd+2ae+2ce+e^2-af-df) \neq 0$

(7) $\Sigma = c^2 - ad - 2cd + 2ae + 2ce + e^2 - af - df \neq 0$, $a + c + e \neq 0$, $e - c - f \neq 0$, $h(cd + df - ce - e^2)\Sigma > 0$, $h(ae - ad - cd)\Sigma > 0$, $h(ce + c^2 + ae - af)\Sigma > 0$, $ce \neq 0$;

(8) $3c - 3d + e \neq 0$, $a + d = 0$, $3a + 3c + e \neq 0$, $dh(3c - 3d + e) < 0$, $h(9cd - 3ce - e^2 + 9df)(3c - 3d + e)(3a + 3c + e) > 0$, $h(3ac + ae + 3c^2 + 3cd + ce + 3df)(3c - 3d + e)(3a + 3c + e) > 0$, $cde(9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df) \neq 0$;

(9) $3c-3d+e = 0$, $a+d \neq 0$, $3c-e+3f \neq 0$, $dh(a+d) < 0$, $h(3ad+3cd-ae)(a+d)(3c-e+3f) > 0$, $h(ac-ad+af-cd)(a+d)(3c-e+3f) > 0$, $cde(9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df) \neq 0$;

(10) $\Sigma_1 = 9c^2 - 9ad - 18cd + 6ae + 6ce + e^2 - 9af - 9df \neq 0$, $a + d \neq 0$, $3c-3d+e \neq 0$, $h(9cd-3ce-e^2+9df)\Sigma_1 > 0$, $h(3c^2+ce+ae-3af)\Sigma_1 > 0$, $h(3ad + 3cd - ae)\Sigma_1 < 0$, $ce \neq 0$;

(11) $3a+c+e = 0$, $c-e+3f \neq 0$, $he(c-e+3f) < 0$, $h(a+d)(c-e+3f)(ce+e^2 - 9df - 3cd) > 0$, $h(a + d)(c - e + 3f)(ce + e^2 + 3ac + 9af) > 0$, $a + d \neq 0$, $ce\big(9ad - c^2 + 6cd - 6ae - 2ce - e^2 + 9af + 9df\big) \neq 0$;

(12) $3a + c + e \neq 0$, $c - e + 3f = 0$, $3d - c - e \neq 0$, $he(3a + c + e) < 0$, $hc(3d - c - e) > 0$, $h(3ad + cd - ae)(3d - c - e)(3a + c + e) > 0$, $ce\big(9ad - c^2 + 6cd - 6ae - 2ce - e^2 + 9af + 9df\big) \neq 0$;

(13) $\Sigma_2 = c^2-9ad-6cd+6ae+2ce+e^2-9af-9df \neq 0$, $h(ae-3ad-cd)\Sigma_2 > 0$, $3a + c + e \neq 0$, $c - e + 3f \neq 0$, $h(3cd - ce - e^2 + 9df)\Sigma_2 > 0$, $h(c^2 + ce + 3ae - 9af)\Sigma_2 > 0$, $ce \neq 0$;

(14) $a + d = 0$, $c - 3d + 3e \neq 0$, $3a + c + 3e \neq 0$, $dh(c - 3d + 3e) < 0$, $h(cd - ce - 3e^2 + 3df)(c - 3d + 3e)(3a + c + 3e) > 0$, $h(3ac + 9ae + c^2 + 3cd + 3ce + 9df)(c - 3d + 3e)(3a + c + 3e) > 0$, $d(2bc - b^2 - c^2 + 9ad + 6cd - 18ae - 6be - 6ce - 9e^2 + 9af + 6bf + 9df)(3bc^2d - 3abce - 4bc^2e + 9acde - 3bcde + 3c^2de - 9abe^2 - 12bce^2 + 9cde^2 + 9bcdf - 9bdef) \neq 0$;

(15) $a+d \neq 0$, $c-3d+3e = 0$, $c-3e+3f \neq 0$, $dh(a+d) < 0$, $h(3ad+cd-3ae)(a+d)(c-3e+3f) > 0$, $h(ac+3af-3ad-cd)(a+d)(c-3e+3f) > 0$, $(-b^2+2bc-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(bc^2d+abce - 3abde + 3acde - 4bcde + c^2de - 3ace^2 + 3bcdf + 3abef) \neq 0$;

(16) $\Sigma_3 = c^2 - 9ad - 6cd + 18ae + 6ce + 9e^2 - 9af - 9df \neq 0$, $a + d \neq 0$, $c-3d+3e \neq 0$, $h(3ae-3ad-cd)\Sigma_3 > 0$, $h(cd-ce-3e^2+3df)\Sigma_3 > 0$, $h(c^2 + 3ce + 9ae - 9af)\Sigma_3 > 0$, $(2bc - b^2 - c^2 + 9ad + 6cd - 18ae - 6be - 6ce - 9e^2 + 9af + 6bf + 9df)(3bc^2d - 4bc^2e + 9acde + 3c^2de - 9abe^2 - 9ace^2 - 12bce^2 + 9bcdf + 9abef) \neq 0$;

(17) $b-c+d-e = 0$, $a+2b+d \neq 0$, $b-c+e-f \neq 0$, $h(b+d)(a+2b+d) < 0$, $h(b^2 - cd + be - bc + ae - ad)(a + 2b + d)(b - c + e - f) > 0$, $h(cd + ab - ac + ad - af - bf)(a + 2b + d)(b - c + e - f) > 0$,

$(b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c-b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef)\neq 0$;

(18) $b-c+d-e\neq 0$, $a+2b+d=0$, $a+b+c+e\neq 0$, $h(b+d)(b-c+d-e)>0$, $h(e^2-cd+ce+be-bf-df)(b-c+d-e)(a+b+c+e)>0$, $h(-ac-ae-cb-c^2-cd-ce-bf-df)(b-c+d-e)(a+b+c+e)>0$, $(2bc-b^2-c^2+ad+2cd-2ae-2be-2ce-e^2+af+2bf+df)(b^2ce-bc^2d+bc^2e+acde+c^2de-abe^2+bce^2+cde^2-b^2cf-bcdf-b^2ef-bdef)\neq 0$;

(19) $\omega=b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df\neq 0$, $b-c+d-e\neq 0$, $a+2b+d\neq 0$, $h(cd-be-ce-e^2+bf+df)\omega>0$, $h(b^2-ad-cd+ae-bc+be)\omega>0$, $h(c^2+ce+ae-af-bc-bf)\omega>0$, $(bc^2d-b^2ce-bc^2e-acde-c^2de+abe^2+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef)\neq 0$;

(20) $3b-c+3d-e=0$, $a+2b+d\neq 0$, $3b-c+e-3f\neq 0$, $h(b+d)(a+2b+d)<0$, $h(a+2b+d)(3b-c+e-3f)(3b^2-cd-bc+be-3ad+ae)>0$, $h(a+2b+d)(3b-c+e-3f)(3ab-ac+3ad+cd-3af-3bf)>0$, $(9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(-3b^3c+b^2c^2-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde+3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef)\neq 0$;

(21) $3b-c+3d-e\neq 0$, $a+2b+d=0$, $3a+3b+c+e\neq 0$, $h(b+d)(3b-c+3d-e)>0$, $h(3a+3b+c+e)(3b-c+3d-e)(3be+ce+e^2-3cd-9bf-9df)>0$, $h(3a+3b+c+e)(3b-c+3d-e)(3ac+3ae+3bc+c^2+3cd+ce+9bf+9df)>0$, $(6bc-9b^2-c^2+9ad+6cd-6ae-6be-2ce-e^2+9af+18bf+9df)(bc^2d+4abce+3b^2ce+bc^2e+3acde+4bcde+c^2de+abe^2+bce^2+cde^2+3b^2cf+3bcdf+3b^2ef+3bdef)\neq 0$;

(22) $\omega_1=9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df\neq 0$, $3b-c+3d-e\neq 0$, $a+2b+d\neq 0$, $h(3cd-3be-ce-e^2+9bf+9df)\omega_1>0$, $h(3b^2-3ad-cd+ae-bc+be)\omega_1>0$, $h(c^2-3bc+3ae+ce-9af-9bf)\omega_1>0$, $(bc^2d-5b^2ce+bc^2e+3acde+c^2de+abe^2-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef)\neq 0$;

(23) $b-3c+3d-e=0$, $3a+2b+3d\neq 0$, $b-3c+e-3f\neq 0$, $h(b^2-9ad-9cd+3ae-3bc+be)(3a+2b+3d)(b-3c+e-3f)>0$, $h(b+3d)(3a+2b+3d)<0$, $h(3cd-bf+ab-3ac+3ad-3af)(3a+2b+3d)(b-3c+e-3f)>0$, $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(b^3c-3b^2c^2+3b^2cd-9bc^2d+3ab^2e-9abce+3bc^2e+9abde+9acde+9c^2de+12bcde-3ace^2-bce^2-3b^2cf-9bcdf-9abef-3b^2ef)\neq 0$;

(24) $b-3c+3d-e\neq 0$, $3a+2b+3d=0$, $3a+b+3c+e\neq 0$, $h(b+3d)(b-3c+3d-e)>0$, $h(b-3c+3d-e)(3a+b+3c+e)(e^2-9cd+be+3ce-3bf-9df)>0$, $h(b-3c+3d-e)(3a+b+3c+e)(3ac+ae+bc+3c^2+3cd+ce+bf+3df)<0$, $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+$

$9c^2de + 3abe^2 + 5bce^2 + 3cde^2 - 3b^2cf - 9bcdf + 3b^2ef + 9bdef) \neq 0$;

(25) $b - 3c + 3d - e \neq 0$, $3a + 2b + 3d \neq 0$, $\omega_2 = b^2 + 9c^2 - 9ad + 6ae + e^2 - 18cd + 6ce - 6bc + 2be - 6bf - 9af - 9df \neq 0$, $h(9cd - be - 3ce - e^2 + 3bf + 9df)\omega_2 > 0$, $h(b^2 - 9ad - 9cd + 3aeb - 3bc + be)\omega_2 > 0$, $h(3c^2 + ce + ae - 3af - bc - bf)\omega_2 > 0$, $(5bc^2e - 3bc^2d - b^2ce + 3acde + 3c^2de + abe^2 - ace^2 + bce^2 - b^2cf - 3bcdf - 3abef - b^2ef) \neq 0$;

(26) $c - b - 3d + 3e = 0$, $3a + 2b + 3d \neq 0$, $b - c + 3e - 3f \neq 0$, $h(b+3d)(3a+2b+3d) < 0$, $h(3a+2b+3d)(b-c+3e-3f)(b^2-bc-9ad-3cd+9ae+3be) > 0$, $h(3a + 2b + 3d)(b - c + 3e - 3f)(cd + ab - ac + 3ad - 3af - bf) > 0$, $(b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce + 9e^2 - 9af - 6bf - 9df)(b^3c - b^2c^2 + 3b^2cd - 3bc^2d + 3ab^2e - 3abce + 4b^2ce - bc^2e + 9abde - 9acde + 12bcde - 3c^2de + 9ace^2 + 3bce^2 - 3b^2cf - 9bcdf - 9abef - 3b^2ef) \neq 0$;

(27) $c - b - 3d + 3e \neq 0$, $3a + 2b + 3d = 0$, $h(b + 3d)(b - c + 3d - 3e) > 0$, $h(b - c + 3d - 3e)(3a + b + c + 3e)(be - bf + ce - cd + 3e^2 - 3df) > 0$, $h(b-c+3d-3e)(3a+b+c+3e)(3ac+bc+c^2+3cd+9ae+3ce+3bf+9df) < 0$, $(2bc - b^2 - c^2 + 9ad + 6cd - 18ae - 6be - 6ce - 9e^2 + 9af + 6bf + 9df)(b^2ce - bc^2d + bc^2e - 3acde - c^2de + 3abe^2 + 3bce^2 - 3cde^2 - b^2cf - 3bcdf + b^2ef + 3bdef) \neq 0$;

(28) $c - b - 3d + 3e \neq 0$, $3a + 2b + 3d \neq 0$, $\omega_3 = b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce + 9e^2 - 9af - 6bf - 9df \neq 0$ $h(cd - be - ce - 3e^2 + bf + 3df)\omega_3 > 0$, $h(b^2 - bc - 9ad - 3cd + 9ae + 3be)\omega_3 > 0$, $h(c^2 - bc + 9ae + 3ce - 9af - 3bf)\omega_3 > 0$, $(bc^2d - b^2ce - bc^2e + 3acde + c^2de - 3abe^2 - 3ace^2 - 5bce^2 + b^2cf + 3bcdf + 3abef + b^2ef) \neq 0$.

Theorem 1 is proved in section 3.

Lemma 2. *The family of periodic orbits of the system* (2) *generated by the periodic solutions* $(r^*, \alpha^*, R^*, \beta^*)$ *of differential system* (15) *at every fixed energy level* $H = \varepsilon h$ *with* $h \in \mathbb{R}$ *is given by*

$$\begin{aligned}
x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\cos t + O(\varepsilon^{3/2}),\\
y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h - r^{*2} - R^{*2}}\cos(\alpha^* - t) + O(\varepsilon^{3/2}),\\
z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\cos(\beta^* - t) + O(\varepsilon^{3/2}),\\
p_x(t,\varepsilon) &= \sqrt{\varepsilon}\quad r^*\sin t + O(\varepsilon^{3/2}),\\
p_y(t,\varepsilon) &= \sqrt{\varepsilon}\quad \sqrt{2h + r^{*2} - R^{*2}}\sin(\alpha^* - t) + O(\varepsilon^{3/2}),\\
p_z(t,\varepsilon) &= \sqrt{\varepsilon}\quad R^*\sin(\beta^* - t) + O(\varepsilon^{3/2}).
\end{aligned} \tag{3}$$

Lemma 3 is proved in the end of section 3.

2. The averaging theory of First Order

In this section we recall the basic results from averaging theory that we shall need for proving Theorem 1.

Consider the differential system

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Furthermore we suppose that the functions $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. We define in D the averaged differential system

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{5}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt. \tag{6}$$

As we shall see under convenient assumptions, the equilibria solutions of the averaged system will provide $T-$periodic solutions of system (4).

Theorem 3. *Consider the two initial value problems* (4) *and* (5). *Assume that*

(i) *the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 and $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$;*

(ii) *the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

(a) *If p is an equilibrium point of the averaged system* (5) *satisfy*

$$\det \left(\frac{\partial f_1}{\partial \mathbf{y}} \right) \bigg|_{\mathbf{y}=p} \neq 0, \tag{7}$$

then there is a $T-$periodic solution $\varphi(t, \varepsilon)$ of system (4) *such that $\varphi(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *The kind of stability or instability of the periodic solution $\varphi(t, \varepsilon)$ coincides with the kind of stability or instability of the equilibrium point p of the averaged system* (5). *The equilibrium point p has the kind of stability behavior of the Poincaré map associated to the periodic solution $\varphi(t, \varepsilon)$.*

For a proof of theorem 3, see theorems 11.5, 11.6 and sections 6.3, 11.8 of Verhulst [12].

In order to apply the averaging theory we need to do some changes of variables in the Hamiltonian differential system (2) in order to write it in the normal form of the averaging theory, i.e. to write it as system (4). Therefore first we do a scaling by a factor $\sqrt{\varepsilon}$ in order to introduce a small parameter $\varepsilon > 0$ in the Hamiltonian system. Second, using some generalized polar coordinates in $\mathbb{R}^6$, and after taking as new independent variable an angle coordinate instead of the time, we get a $2\pi-$periodic differential system. Third, fixing the energy level and omitting a redundant variable in every energy level, we will obtain a differential system written in the normal form for applying the averaging theorem (Theorem 3), and finally we shall prove the existence of some isolated periodic solutions in every positive energy level.

3. Proof of theorem 1

We do a scaling using a small parameter $\varepsilon > 0$. In fact, we change in the Hamiltonian system (2) the variables (x, y, z, p_x, p_y, p_z) by $(X, Y, Z\ p_X, p_Y, p_Z)$ where

$x = \sqrt{\varepsilon}\, X$, $y = \sqrt{\varepsilon}\, Y$, $z = \sqrt{\varepsilon}\, Z$, $p_x = \sqrt{\varepsilon}\, p_X$, $p_y = \sqrt{\varepsilon}\, p_Y$ and $p_z = \sqrt{\varepsilon}\, p_Z$. In the new variables, system (2) becomes

$$
\begin{array}{rcl}
\dot{X} &=& -p_X, \\
\dot{Y} &=& p_Y, \\
\dot{Z} &=& p_Z, \\
\dot{p}_X &=& X - \varepsilon X(aX^2 + bY^2 + cZ^2), \\
\dot{p}_Y &=& -Y - \varepsilon Y(bX^2 + dY^2 + eZ^2), \\
\dot{p}_Z &=& -Z - \varepsilon Z(cX^2 + eY^2 + fZ^2).
\end{array} \tag{8}
$$

This differential system again is Hamiltonian with Hamiltonian

$$
\begin{array}{rl}
H = & \frac{\varepsilon}{2}\Big(-p_X^2 + p_Y^2 + p_Z^2 - X^2 + Y^2 + Z^2\Big) + \frac{\varepsilon^2}{4}\Big(aX^4 + 2bX^2Y^2 \\
& +2cX^2Z^2 + dY^4 + 2eY^2Z^2 + fZ^4\Big).
\end{array} \tag{9}
$$

The original and the transformed systems (2) and (8) have the same topological phase portrait because the change of variables is only a scale transformation for all $\varepsilon > 0$, and also system (8) for ε sufficiently small is close to an integrable one.

The periodicity in the independent variable of the differential system is needed to apply the averaging theory, so we change the Hamiltonian (9) and its equations of motion (8) to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in $\mathbb{R}^6$. Defined by

$$
\begin{array}{lll}
X = r\cos\theta, & Y = \rho\cos(\alpha - \theta), & Z = R\cos(\beta - \theta), \\
p_X = r\sin\theta, & p_Y = \rho\sin(\alpha - \theta), & p_Z = R\sin(\beta - \theta).
\end{array} \tag{10}
$$

Of course in this change of variables $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H in the new variables becomes

$$
\begin{array}{rr}
H = & \frac{\varepsilon}{2}(\rho^2 + R^2 - r^2) + \frac{\varepsilon^2}{4}\big[ar^4\cos^4\theta + 2br^2\rho^2\cos^2\theta \\
& \cos^2(\alpha - \theta) + d\rho^4\cos^4(\alpha - \theta) + 2R^2\big(cr^2\cos^2\theta \\
& +e\rho^2\cos^2(\alpha - \theta)\big)\cos^2(\beta - \theta) + fR^4\cos^4(\beta - \theta)\big],
\end{array} \tag{11}
$$

and the equations of motion (8) become

$$
\begin{aligned}
\dot r &= -\varepsilon r\cos\theta\sin\theta\big[a\,r^2\cos^2\theta+b\,\rho^2\cos^2(\alpha-\theta)\\
&\quad +c\,R^2\cos^2(\beta-\theta)\big],\\
\dot\theta &= 1-\varepsilon\cos^2\theta\big[a\,r^2\cos^2\theta+b\rho^2\cos^2(\alpha-\theta)\\
&\quad +cR^2\cos^2(\beta-\theta)\big],\\
\dot\rho &= -\varepsilon\rho\cos(\alpha-\theta)\sin(\alpha-\theta)\big[b\,r^2\cos^2\theta\\
&\quad +d\rho^2\cos^2(\alpha-\theta)+eR^2\cos^2(\beta-\theta)\big],\\
\dot\alpha &= \varepsilon\Big[-a\,r^2\cos^4\theta-\cos^2\theta\Big(b(r^2+\rho^2)\cos^2(\alpha-\theta)\\
&\quad +cR^2\cos^2(\beta-\theta)\Big)-\cos^2(\alpha-\theta)\Big(d\rho^2\cos^2(\alpha-\theta)\\
&\quad +eR^2\cos^2(\beta-\theta)\Big)\Big],\\
\dot R &= -\varepsilon R\cos(\beta-\theta)\sin(\beta-\theta)\big[c\,r^2\cos^2\theta\\
&\quad +e\rho^2\cos^2(\alpha-\theta)+fR^2\cos^2(\beta-\theta)\big],\\
\dot\beta &= \varepsilon\Big[-a\,r^2\cos^4\theta-b\rho^2\cos^2\theta\cos^2(\alpha-\theta)\\
&\quad -\Big(c(r^2+R^2)\cos^2\theta-e\rho^2\cos^2(\alpha-\theta)\Big)\\
&\quad \cos^2(\beta-\theta)-fR^2\cos^4(\beta-\theta)\Big].
\end{aligned}
\tag{12}
$$

We note that in this system if we take the variable θ as the new independent variable instead of t, we obtain the necessary periodicity for writing the system in the normal form of the averaging theory. In what follows the independent variable will be θ. This means that the new differential system will have only five equations. We denote by a prime the derivative with respect to θ and we expand system (12) in Taylor series in ε. Thus, system (12) becomes

$$
\begin{aligned}
r' &= -\varepsilon\, r\sin\theta\cos\theta\big[a\,r^2\cos^2\theta+b\rho^2\cos^2(\alpha-\theta)\\
&\quad +cR^2\cos^2(\beta-\theta)\big]+O(\varepsilon^2),\\
\rho' &= -\varepsilon\rho\cos(\alpha-\theta)\sin(\alpha-\theta)\big[b\,r^2\cos^2\theta+d\rho^2\cos^2(\alpha-\theta)\\
&\quad +eR^2\cos^2(\beta-\theta)\big]+O(\varepsilon^2),\\
\alpha' &= \varepsilon\big[-\;ar^2\cos^4\theta-d\rho^2\cos^4(\alpha-\theta)\\
&\quad -eR^2\cos^2(\alpha-\theta)\cos^2(\beta-\theta)-\cos^2\theta\big(b(r^2+\rho^2)\\
&\quad \cos^2(\alpha-\theta)+cR^2\cos^2(\beta-\theta)\big)\big]+O(\varepsilon^2),\\
R' &= -\varepsilon\, R\sin(\beta-\theta)\cos(\beta-\theta)\big[c\,r^2\cos^2\theta\\
&\quad +e\rho^2\cos^2(\alpha-\theta)+fR^2\cos^2(\beta-\theta)\big]+O(\varepsilon^2),\\
\beta' &= \varepsilon\big[-fR^2\cos^4(\beta-\theta)-\cos^2\theta\,\big(ar^2\cos^2\theta\\
&\quad +b\rho^2\cos^2(\alpha-\theta)\big)-\cos^2(\beta-\theta)\big(c(r^2+R^2)\cos^2\theta\\
&\quad +e\rho^2\cos^2(\alpha-\theta)\big)\big]+O(\varepsilon^2).
\end{aligned}
\tag{13}
$$

System (13) is 2π-periodic respect to the variable θ, i.e. it is written as the normal form (4) but it is not ready for applying the averaging theory, we must fix the value of the first integral $H=\varepsilon h$ with $h\in\mathbb{R}$, otherwise the Jacobian (7) will be zero because the periodic orbits are non-isolated leaving on cylinders parameterized by the energy, see for more details [1].

Solving ρ from $H = \varepsilon h$, we get two positive solutions, but the unique with physical meaning expanded in Taylor series in ε is

$$\rho = \sqrt{2h + r^2 - R^2} + O(\varepsilon). \tag{14}$$

Since $\rho \geq 0$ we need that $2h + r^2 - R^2 \geq 0$.

Substituting ρ in system (13) we get the differential system

$$\begin{aligned}
r' &= -\varepsilon\, r \sin\theta \cos\theta \big[a\, r^2 \cos^2\theta + b(2h + r^2 - R^2)\cos^2(\alpha - \theta) \\
&\quad + cR^2 \cos^2(\beta - \theta)\big] + O(\varepsilon^2), \\
\alpha' &= \varepsilon \Big[-\ ar^2 \cos^4\theta + d(R^2 - 2h - r^2)\cos^4(\alpha - \theta) \\
&\quad - eR^2 \cos^2(\alpha - \theta)\cos^2(\beta - \theta) + \cos^2\theta \Big(b(R^2 - 2h - 2r^2) \\
&\quad \cos^2(\alpha - \theta) - cR^2 \cos^2(\beta - \theta)\Big)\Big] + O(\varepsilon^2), \\
R' &= -\varepsilon\, R \sin(\beta - \theta)\cos(\beta - \theta)\big[c\, r^2 \cos^2\theta + e(2h + r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) + fR^2 \cos^2(\beta - \theta)\big] + O(\varepsilon^2), \\
\beta' &= \varepsilon \Big[-\ ar^2 \cos^4\theta + \cos^2(\beta - \theta)(-e(2h + r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) - fR^2 \cos^2(\beta - \theta)) - \cos^2\theta \Big(b(2h + r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) + c(R^2 + r^2)\cos^2(\beta - \theta)\Big)\Big] + O(\varepsilon^2).
\end{aligned} \tag{15}$$

Following the notation of the averaging theory given in section 2, the function $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ of (4) is

$$\begin{aligned}
F_{11} &= -r \sin\theta \cos\theta \big[a\, r^2 \cos^2\theta + b(2h + r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) + cR^2 \cos^2(\beta - \theta)\big], \\
F_{12} &= -\ ar^2 \cos^4\theta - d(2h + r^2 - R^2)\cos^4(\alpha - \theta) \\
&\quad - eR^2 \cos^2(\alpha - \theta)\cos^2(\beta - \theta) - \cos^2\theta \big(b(2h + 2r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) + cR^2 \cos^2(\beta - \theta)\big), \\
F_{13} &= -R \sin(\beta - \theta)\cos(\beta - \theta)\big[c\, r^2 \cos^2\theta + e(2h + r^2 - R^2) \\
&\quad \cos^2(\alpha - \theta) + fR^2 \cos^2(\beta - \theta)\big], \\
F_{14} &= -\ ar^2 \cos^4\theta - e(2h + r^2 - R^2)\cos^2(\alpha - \theta)\cos^2(\beta - \theta) \\
&\quad - fR^2 \cos^4(\beta - \theta) - \cos^2\theta \big(b(2h + r^2 - R^2)\cos^2(\alpha - \theta) \\
&\quad + c(R^2 + r^2)\cos^2(\beta - \theta)\big),
\end{aligned} \tag{16}$$

where $F_{1j} = F_{1j}(\theta, r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

From (6) and (16) we compute the averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ and we obtain

$$\begin{aligned}
f_{11} = \;& -\frac{1}{8}r\big[b(2h + r^2 - R^2)\sin 2\alpha + cR^2 \sin 2\beta\big],\\
f_{12} = \;& \frac{1}{8}\big[-6dh - 3ar^2 - 3dr^2 + 3dR^2 + (-2bh - 2br^2 + bR^2)\\
& (2 + \cos 2\alpha) - eR^2(2 + \cos 2(\alpha - \beta)) - cR^2(2 + \cos 2\beta)\big],\\
f_{13} = \;& \frac{1}{8}R\big[e(2h + r^2 - R^2)\sin 2(\alpha - \beta) - cr^2 \sin 2\beta\big],\\
f_{14} = \;& \frac{1}{8}\Big[-3ar^2 - 3fR^2 - (2h + r^2 - R^2)\Big(2(b + e) + b\cos 2\alpha\\
& + e\cos 2(\alpha - \beta)\Big) - c(r^2 + R^2)(2 + \cos 2\beta)\Big],
\end{aligned}$$

where $f_{1j} = f_{1j}(r, \alpha, R, \beta)$ for $j = 1, 2, 3, 4$.

According to theorem 3 our aim is to find the zeros $(r^*, \alpha^*, R^*, \beta^*)$ of

$$f_{1i}(r, \alpha, R, \beta) = 0 \quad \text{for } i = 1, 2, 3, 4, \tag{17}$$

and after we must check that the Jacobian determinant (7) evaluated in these zeros are different from zero.

Then from $f_{11}(r, \alpha, R, \beta) = 0$ we obtain either $r = 0$, or $b\sin 2\alpha \neq 0$ and $r = \sqrt{R^2 - 2h - \dfrac{cR^2 \sin 2\beta}{b \sin 2\alpha}}$, or $b\sin 2\alpha = 0$.

Case 1: $r = 0$. Substituting r in f_{1i}, for $i = 2, 3, 4$, we get

$$\begin{aligned}
f_{12}(0, \alpha, R, \beta) = \;& \frac{1}{8}\big[-6dh + 3dR^2 + (-2bh + bR^2)(2 + \cos 2\alpha) - eR^2\\
& (2 + \cos 2(\alpha - \beta)) - cR^2(2 + \cos 2\beta)\big],\\
f_{13}(0, \alpha, R, \beta) = \;& -\frac{1}{8}eR(-2h + R^2)\sin 2(\alpha - \beta),\\
f_{14}(0, \alpha, R, \beta) = \;& \frac{1}{8}\big[-4(b + e)h + (2b - 2(c - e) - 3f)R^2 -\\
& (2h - R^2)\Big(b\cos 2\alpha + e\cos 2(\alpha - \beta)\Big) - cR^2\cos 2\beta\big].
\end{aligned}$$

From $f_{13}(0, \alpha, R, \beta) = 0$ we have the following four subcases: $e = 0$, $R = 0$, $R = \sqrt{2h}$, $\alpha = \beta + \dfrac{k\pi}{2}$ with $k \in \mathbb{Z}$.

Subcase 1.1: $e = 0$. The Jacobian is zero in this subcase because f_{13} vanish and the averaging theory cannot provide information about the periodic orbits. Hence, *in what follows in case* 1 *we assume that* $e \neq 0$.

Subcase 1.2: When $R = 0$, f_{12} and f_{14} become

$$\begin{aligned}
f_{12}(0, \alpha, 0, \beta) = \;& -\frac{1}{4}h\big[2b + 3d + b\cos 2\alpha\big],\\
f_{14}(0, \alpha, 0, \beta) = \;& -\frac{1}{4}h\big[2b + 2e + b\cos 2\alpha + e\cos 2(\alpha - \beta)\big].
\end{aligned}$$

If $b \neq 0$ then $f_{12}(0, \alpha, 0, \beta) = 0$ implies $\alpha_{\pm} = \pm \frac{1}{2} \arccos - \frac{3d+2b}{b}$. Substituting α_+ in $f_{14}(0, \alpha, R, \beta)$ we get

$$f_{14}(0, \alpha, 0, \beta) = \frac{1}{4} h \left[e \cos \left(2\beta + \arccos \frac{3d+2b}{b} \right) + 3d - 2e \right].$$

Hence $f_{14}(0, \alpha, 0, \beta) = 0$ implies $\beta_{+\pm} = \frac{1}{2} \big[- \arccos \frac{3d+2b}{b} \pm \arccos \frac{2e-3d}{e} \big]$ if $e \neq 0$.

Substituting α_- in $f_{14}(0, \alpha, 0, \beta)$ we get

$$f_{14}(0, \alpha, 0, \beta) = \frac{1}{4} h \left[e \cos \left(2\beta - \arccos \frac{3d+2b}{b} \right) + 3d - 2e \right].$$

Therefore $f_{14}(0, \alpha, 0, \beta) = 0$ implies $\beta_{-\pm} = \frac{1}{2} \big[\arccos \frac{3d+2b}{b} \pm \arccos \frac{2e-3d}{e} \big]$ if $e \neq 0$.

Supposing that

$$h > 0, \quad eb \neq 0, \quad \left| \frac{3d+2b}{b} \right| < 1 \quad \text{and} \quad \left| \frac{2e-3d}{e} \right| < 1. \tag{18}$$

System (17) has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{2h}$ given by

$$\begin{aligned} &\left(0, \frac{1}{2} \arccos - \frac{3d+2b}{b}, 0, \frac{1}{2} \Big[- \arccos \frac{3d+2b}{b} \pm \arccos \frac{2e-3d}{e} \Big] \right), \\ &\left(0, -\frac{1}{2} \arccos - \frac{3d+2b}{b}, 0, \frac{1}{2} \Big[\arccos \frac{3d+2b}{b} \pm \arccos \frac{2e-3d}{e} \Big] \right), \end{aligned} \tag{19}$$

which reduce to two solutions if either $\left| \frac{3d+2b}{b} \right| = 1$, $\left| \frac{2e-3d}{e} \right| < 1$, $h > 0$ and $eb \neq 0$, or $|(3d+2b)/b| < 1$, $|(2e-3d)/e| = 1$, $h > 0$ and $eb \neq 0$. We have one solution if $|(3d+2b)/b| = |(2e-3d)/e| = 1$, $h > 0$ and $eb \neq 0$.

Now we check if the Jacobian of f_1 evaluated in these solutions is different from zero. By definition the Jacobian is

$$J_{f_1} = \quad |D_{r\alpha R\beta} f_1(S^*)| = \left| \begin{matrix} \frac{\partial f_{11}}{\partial r} & \frac{\partial f_{11}}{\partial \alpha} & \frac{\partial f_{11}}{\partial R} & \frac{\partial f_{11}}{\partial \beta} \\ \frac{\partial f_{12}}{\partial r} & \frac{\partial f_{12}}{\partial \alpha} & \frac{\partial f_{12}}{\partial R} & \frac{\partial f_{12}}{\partial \beta} \\ \frac{\partial f_{13}}{\partial r} & \frac{\partial f_{13}}{\partial \alpha} & \frac{\partial f_{13}}{\partial R} & \frac{\partial f_{13}}{\partial \beta} \\ \frac{\partial f_{14}}{\partial r} & \frac{\partial f_{14}}{\partial \alpha} & \frac{\partial f_{14}}{\partial R} & \frac{\partial f_{14}}{\partial \beta} \end{matrix} \right|_{(r,\alpha,R,\beta)=S^*}.$$

So $J_{f_1(S^*)} = \frac{3}{64} h^4 e^2 (b+3d)(b+d) \left[1 - \left(\frac{2e-3d}{e} \right)^2 \right]$.

Assuming that $(b+d)(b+3d) \neq 0$ and (18) hold. This supposition is not empty because the value $b = 1, d = -2/3, e = -4/3, h = 1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (19) of system (17) provide only two periodic solutions of differential system (15) because when $R = 0$ the two solutions of β provide the same initial conditions in (10). In the remaining subcases in what follows of all this paper we have two possibilities and in both of them we do not have results. The first one when the $J_{f_1(S^*)} = 0$, therefore the averaging theory does not provide information about the periodic solution. The second one when the set of conditions on the parameters which guarantee the existence of the solutions is empty.

If $b = 0$ then $f_{12} = -3dh/4 =$ constant. The Jacobian will be zero and the averaging theory does not give results.

If $b \neq 0$ and $e = 0$ then $f_{14} = 3dh/4 =$ constant. The Jacobian will be zero and the averaging theory does not give results.

Subcase 1.3: $R = \sqrt{2h}$ so $h > 0$. Then f_{12} and f_{14} become

$$\begin{aligned} f_{12}(0, \alpha, \sqrt{2h}, \beta) = & \ -\frac{1}{4}h\big[2e + 2c + c\cos 2\beta + e\cos 2(\alpha - \beta)\big], \\ f_{14}(0, \alpha, \sqrt{2h}, \beta) = & \ -\frac{1}{4}h\big[2c + 3f + c\cos 2\beta\big]. \end{aligned}$$

If $c \neq 0$, solving $f_{14} = 0$ we get $\beta_\pm = \pm\frac{1}{2}\arccos -\frac{3f+2c}{c}$. Substituting $\beta_\pm$ respectively in f_{12} we have

$$f_{12}(0, \alpha, \sqrt{2h}, \beta_\pm) = \frac{h}{4}\left[-2e + 3f + e\cos\left(2\alpha \pm \arccos\frac{3f+2c}{c}\right)\right].$$

Solving $f_{12} = 0$ with respect to α, if $e \neq 0$ we obtain the four solutions

$$\alpha_{\pm\mp} = \frac{1}{2}\left[\pm\arccos\frac{2e-3f}{e} \mp \arccos\frac{3f+2c}{c}\right].$$

Assuming

$$h > 0, \quad ec \neq 0, \quad \left|\frac{2e-3f}{e}\right| < 1 \quad \text{and} \quad \left|\frac{3f+2c}{c}\right| < 1. \tag{20}$$

System (17) has four solutions $S^* = (r, \alpha, R, \beta)$ with $\rho = 0$ given by

$$\begin{aligned} &\left(0, \pm\frac{1}{2}\arccos\frac{2e-3f}{e} - \frac{1}{2}\arccos\frac{3f+2c}{c}, \sqrt{2h}, \frac{1}{2}\arccos -\frac{3f+2c}{c}\right), \\ &\left(0, \pm\frac{1}{2}\arccos\frac{2e-3f}{e} + \frac{1}{2}\arccos\frac{3f+2c}{c}, \sqrt{2h}, -\frac{1}{2}\arccos -\frac{3f+2c}{c}\right), \end{aligned} \tag{21}$$

which reduce to two solutions if either $h > 0$, $|(2e-3f)/e| < 1$, $ec \neq 0$ and $|(3f+2c)/c| = 1$ or $|(2e-3f)/e| = 1$ and $|(3f+2c)/c| < 1$. We have one solution if $h > 0$, $ec \neq 0$ and $|(2e-3f)/e| = |(3f+2c)/c| = 1$.

Computing the Jacobian on these solutions we get

$$J_{f_1(S^*)} = \frac{3}{32}h^4e^2(c+3f)(c+f)\left[1 - \left(\frac{2e-3f}{e}\right)^2\right].$$

Supposing that $(c+3f)(c+f) \neq 0$ and (20) hold. This assumption is not empty because it is satisfied for the value $c = 1, f = -3/4, e = -7/4, h = 1$. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (21) of system (17) provide only two periodic solutions of differential system (15) because since $\rho = 0$ the two solutions of α provide the same initial conditions in (10).

If $c = 0$ then $f_{14} = -3fh/4 =$ constant.

If $c \neq 0$ and $e = 0$ then $f_{12} = 3fh/4$= constant.

Subcase 1.4: $\alpha = \beta + \dfrac{k\pi}{2}$. We consider four subcases $k = 0, 1, 2, 3$.

Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$ together.

Subcase 1.4.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\alpha = \beta$ or $\alpha = \beta + \pi$. Then

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) = & -\frac{1}{8}\big[2h(2b + 3d + b\cos 2\beta) - \big(2b - 2c + 3d - 3e + \\ & (b - c)\cos 2\beta\big)R^2\big], \\ f_{14}(0, \alpha, R, \beta) = & -\frac{1}{8}\big[3fR^2 + cR^2(2 + \cos 2\beta) + (2h - R^2) \\ & (2b + 3e + b\cos 2\beta)\big]. \end{aligned}$$

Subcase 1.4.1.1: $D_1 = 2b - 2c + 3d - 3e + (b - c)\cos 2\beta \neq 0$. Solving $f_{12} = 0$ we get $R = \sqrt{2h\,(2b + 3d + b\cos 2\beta)\,/D_1}$. Substituting R in f_{14} and solving $f_{14} = 0$ with respect to the variable β we get, if $cd - be - ce + bf \neq 0$, $\beta = \pm\dfrac{1}{2}\arccos[(-2cd + 2be + 2ce + 3e^2 - 2bf - 3df)/(cd - be - ce + bf)]$. Let $d - 2e + f \neq 0$, $R = \sqrt{2h(d - e)/(d - 2e + f)}$ and $\rho = \sqrt{2h(f - e)/(d - 2e + f)}$.

With the condition that

$$\begin{aligned} cd - be - ce + bf \neq 0, \quad (d - 2e + f) \neq 0, \quad h(d - 2e + f)(f - e) > 0, \\ h(d - 2e + f)(d - e) > 0 \quad \text{and} \quad \big|\Delta_1\big| < 1, \end{aligned} \tag{22}$$

where $\Delta_1 = (-2cd + 2be + 2ce + 3e^2 - 2bf - 3df)/(cd - be - ce + bf)$, system (17) for $k = 0$ and $k = 2$ has four solutions $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2h(f - e)}{(d - 2e + f)}}$ given by

$$\begin{aligned} & \left(0, \beta, \sqrt{\frac{2h(d - e)}{(d - 2e + f)}}, \beta = \pm\frac{1}{2}\arccos\Delta_1\right), \\ & \left(0, \beta + \pi, \sqrt{\frac{2h(d - e)}{(d - 2e + f)}}, \beta = \pm\frac{1}{2}\arccos\Delta_1\right), \end{aligned} \tag{23}$$

which reduce to two solutions if $cd - be - ce + bf \neq 0$, $(d - 2e + f) \neq 0$, $h(d - 2e + f)(f - e) > 0$, $h(d - 2e + f)(d - e) > 0$ and $\big|\Delta_1\big| = 1$.

Its Jacobian

$$\begin{aligned} J_{f_1(S^*)} = \ & \frac{9eh^4}{32(d - 2e + f)^3}(f - e)(d - e)(-be + bf + cd - ce + 3df - 3e^2) \\ & (-be + bf + cd - ce + df - e^2). \end{aligned}$$

Assuming that $e(-be + bf + cd - ce + 3df - 3e^2)(-be + bf + cd - ce + df - e^2) \neq 0$ and (22) hold. This supposition is satisfied when $b = 5, c = 0, d = 0, f = -1/2$, $e = -1, h = 1$. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (23) of system (17) provide four periodic solutions of differential system (15).

If $cd - be - ce + bf = 0$ we get $f_{14} = 3h(2be + 3e^2 - 2cd + 2ce - 2bf - 3df)/4D_1$. So either f_{14} never is zero, or f_{14} is identically zero. In both cases the averaging theory does not provide information.

Subcase 1.4.1.2: $D_1 = 0$. We have

$$\begin{aligned} f_{12}(0,\alpha,R,\beta) = & -\frac{1}{4}h\big[2b+3d+b\cos 2\beta\big], \\ f_{14}(0,\alpha,R,\beta) = & -\frac{1}{4}h\big[2b+3e+b\cos 2\beta+2R^2\big(2b-2c+3e-3f \\ & +(b-c)\cos 2\beta\big)\big]. \end{aligned}$$

If $b \neq 0$, solving $f_{12} = 0$ we have $\beta = \pm\frac{1}{2}\arccos -\frac{3d+2b}{b}$. Substituting β in f_{14} and solving $f_{14} = 0$ we get if $bd-cd-be+bf \neq 0$, $R = \sqrt{\dfrac{2hb(d-e)}{bd-cd-be+bf}}$. Then $\rho = \sqrt{\dfrac{2h(cd-bf)}{bd-cd-be+bf}}$.

In case that

$$\begin{aligned} & bd-cd-be+bf \neq 0, \quad b \neq 0, \quad \left|\frac{3d+2b}{b}\right| < 1, \\ & h(bd-cd-be+bf)(cd-bf) > 0 \quad \text{and} \\ & hb(bd-cd-be+bf)(d-e) > 0, \end{aligned} \tag{24}$$

system (17) for $k = 0$ and $k = 2$ has four solutions $S^* = (r,\alpha,R,\beta)$ with $\rho = \sqrt{\dfrac{2h(cd-bf)}{(bd-cd-be+bf)}}$ given by

$$\begin{aligned} & \left(0,\beta,\sqrt{\frac{2hb(d-e)}{bd-cd-be+bf}},\beta = \pm\frac{1}{2}\arccos -\frac{3d+2b}{b}\right), \\ & \left(0,\beta+\pi,\sqrt{\frac{2hb(d-e)}{bd-cd-be+bf}},\beta = \pm\frac{1}{2}\arccos -\frac{3d+2b}{b}\right), \end{aligned} \tag{25}$$

which reduce to two solutions if $bd-cd-be+bf \neq 0$, $b \neq 0$, $\left|\dfrac{3d+2b}{b}\right| = 1$, $h(bd-cd-be+bf)(cd-bf) > 0$ and $hb(bd-cd-be+bf)(d-e) > 0$.

$J_{f_1(S^*)} = \dfrac{9beh^4}{32(bd-cd-be+bf)^4}\big[(b+d)(b+3d)(d-e)(d-2e+f)(bf-cd)(ce-bf)^2\big]$.

Assuming that $e(b+d)(b+3d)(d-2e+f)(ce-bf) \neq 0$ and (24) hold. This supposition is not empty because the value $b=3, c=5, d=-2, f=-3$, $e=-1, h=1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (25) of system (17) provide four periodic solutions of differential system (15).

If $b = 0$ we have $f_{12} = -3dh/4$=constant.

If $b \neq 0$ and $bd-cd-be+bf = 0$ we get $f_{14} = 3(d-e)h/4$=constant.

Subcase 1.4.2: Assume that either $k=1$ or $k=3$, i.e. either $\alpha=\beta+\dfrac{\pi}{2}$ or $\alpha=\beta+\dfrac{3\pi}{2}$ then

$$\begin{aligned} f_{12}(0,\alpha,R,\beta)= & -\frac{1}{8}\big[2h(2b+3d+b\cos 2\beta)+(2c-2b-3d+e+\\ & (b+c)\cos 2\beta)R^2\big],\\ f_{14}(0,\alpha,R,\beta)= & \frac{1}{8}\big[-2(2b+e)h+(2b-2c+e-3f)R^2+\big(2bh-\\ & (b+c)R^2\big)\cos 2\beta\big]. \end{aligned}$$

Subcase 1.4.2.1: $D_2=2c-2b-3d+e+b\cos 2\beta+c\cos 2\beta\neq 0$. So solving $f_{12}(0,\alpha,R,\beta)=0$ we get $R=\sqrt{2h\left(-2b-3d-b\cos 2\beta\right)/D_2}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to the variable β we obtain, if $3cd+be-ce-3bf\neq 0$, $\beta=\pm\dfrac{1}{2}\arccos\Delta_2$ where $\Delta_2=\dfrac{2be+2ce+e^2-6cd-6bf-9df}{3cd+be-ce-3bf}$. Then $R=\sqrt{\dfrac{2h(3d-e)}{3d-2e+3f}}$ and $\rho=\sqrt{\dfrac{2h(3f-e)}{3d-2e+3f}}$.

Conceding that

$$\begin{aligned} & 3cd+be-ce-3bf\neq 0, 3d-2e+3f\neq 0,\\ & h(3f-e)(3d-2e+3f)>0,\quad h(3d-e)(3d-2e+3f)>0\\ & \quad\text{and}\quad |\Delta_2|<1. \end{aligned}\tag{26}$$

System (17) for $k=1$ or $k=3$ has four solutions $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(3f-e)}{3d-2e+3f}}$ given by

$$\begin{aligned} &\left(0,\beta+\frac{\pi}{2},\sqrt{\frac{2h(3d-e)}{3d-2e+3f}},\beta=\pm\frac{1}{2}\arccos\Delta_2\right),\\ &\left(0,\beta+\frac{3\pi}{2},\sqrt{\frac{2h(3d-e)}{3d-2e+3f}},\beta=\pm\frac{1}{2}\arccos\Delta_2\right), \end{aligned}\tag{27}$$

which reduce to two solutions if $3cd+be-ce-3bf\neq 0$, $3d-2e+3f\neq 0$, $h(3f-e)(3d-2e+3f)>0$, $h(3d-e)(3d-2e+3f)>0$ and $|\Delta_2|=1$.

The Jacobian is

$$\begin{aligned} J_{f_1(S^*)}= & \frac{eh^4(3f-e)(3d-e)}{32(3d-2e+3f)^3}(-9cd+be+3ce+e^2-3bf-9df)\\ & (3cd-3be-ce-e^2+9bf+9df). \end{aligned}$$

Assuming that $(-9cd+be+3ce+e^2-3bf-9df)(3cd-3be-ce-e^2+9bf+9df)\neq 0$ and (26) hold. This supposition is not empty because the value $b=5/8, c=1$, $d=-1, f=-1, e=0, h=1$ satisfy it. Therefore $J_{f_1(S^*)}\neq 0$ and the four solutions (27) of system (17) provide four periodic solutions of differential system (15).

When $3d-2e+3f\neq$ and $3cd+be-ce-3bf=0$ we obtain $f_{14}=h(6cd-2be-2ce-e^2+6bf+9df)/(4D_2)$. So either f_{14} never is zero, or f_{14} is identically zero. In both cases the averaging theory does not provide information.

Subcase 1.4.2.2: $D_2=0$. Then $f_{12}=\dfrac{h}{4}(b\cos 2\beta-2b-3d)$. Solving $f_{12}=0$ when $b\neq 0$ we get $\beta=\pm\dfrac{1}{2}\arccos(3d+2b)/b$. Substituting β in f_{14} and solving

$f_{14} = 0$ if $4bc+3bd+3cd-be+3bf \neq 0$ we obtain $R = \sqrt{\dfrac{2hb(3d-e)}{4bc+3bd+3cd-be+3bf}}$ and $\rho = \sqrt{\dfrac{2h(4bc+3cd+3bf)}{4bc+3bd+3cd-be+3bf}}$.

Assuming that

$$
\begin{array}{l}
4bc+3bd+3cd-be+3bf \neq 0, \quad b \neq 0, \\
hb(4bc+3bd+3cd-be+3bf)(3d-e) > 0, \\
h(4bc+3bd+3cd-be+3bf)(4bc+3cd+3bf) > 0, \\
\text{and} \quad \left|\dfrac{3d+2b}{b}\right| < 1.
\end{array} \tag{28}
$$

System (17) for $k=1$ and $k=3$ has four solutions $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2h(4bc+3cd+3bf)}{4bc+3bd+3cd-be+3bf}}$ given by

$$
\begin{array}{l}
\left(0, \beta + \dfrac{\pi}{2}, \sqrt{\dfrac{2hb(3d-e)}{4bc+3bd+3cd-be+3bf}}, \beta = \pm\dfrac{1}{2}\arccos\dfrac{2b+3d}{b}\right), \\
\left(0, \beta + \dfrac{3\pi}{2}, \sqrt{\dfrac{2hb(3d-e)}{4bc+3bd+3cd-be+3bf}}, \beta = \pm\dfrac{1}{2}\arccos\dfrac{2b+3d}{b}\right),
\end{array} \tag{29}
$$

which reduce to two solutions when $4bc+3bd+3cd-be+3bf \neq 0$, $b \neq 0$, $h(4bc+3bd+3cd-be+3bf)(4bc+3cd+3bf) > 0$, $h(4bc+3bd+3cd-be+3bf)2hb(3d-e) > 0$ and $\left|\dfrac{2b+3d}{b}\right| = 1$.

We have

$$
\begin{aligned}
J_{f_1(S^*)} = \; & -\frac{3beh^4(b+d)(b+3d)}{32(4bc+3bd+3cd-be+3bf)^4}(3d-e)(3d-2e+3f)(4bc+3cd \\
& +3bf)(4bc+ce+3bf)^2.
\end{aligned}
$$

In case that $e(b+d)(b+3d)(3d-2e+3f)(4bc+ce+3bf) \neq 0$ and (28) hold, this supposition is true for the value $b=2, c=8, d=-1, f=-6, e=-9, h=1$. Then we have $J_{f_1(S^*)} \neq 0$ and the four solutions (29) of system (17) provide four periodic solutions of differential system (15).

If $b=0$, $f_{12} = -3dh/4$=constant.

If $b \neq 0$ and $4bc+3bd+3cd-be+3bf = 0$ we get $f_{14} = h(3d-e)/(4b)$=constant.

Case 2: $b\sin 2\alpha \neq 0$**,** $r=\sqrt{R^2-2h-\dfrac{cR^2\sin 2\beta}{b\sin 2\alpha}}$. Then

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = \;& \frac{1}{32b}\big[4b(6ah+4bh-(3a+2(b+c+e))R^2\\
&+b(2h-R^2)\cos 2\alpha-4bR^2(c+e\cos 2\alpha)\cos 2\beta\\
&+\frac{R^2\sin 2\beta}{\sin\alpha\cos\alpha}(6ac+8bc+6cd-be+4bc\cos 2\alpha\\
&+be\cos 4\alpha)\big],\\
f_{13}(r,\alpha,R,\beta) = \;& \frac{cR\sin 2\beta}{4}\big[h-\frac{R^2}{2b\sin 2\alpha}[b\sin 2\alpha-c\sin 2\beta\\
&+e\sin 2(\alpha-\beta)]\big],\\
f_{14}(r,\alpha,R,\beta) = \;& \frac{1}{32b}\Bigg[8b(3a+2c)h-2R^2(6ab+8bc-ce+6bf)\\
&+8bc(h-R^2)\cos 2\beta+cR^2\Big[-2e\cos 4\beta\\
&+\frac{1}{\sin\alpha\cos\alpha}\big[(c+e\cos 2\alpha)\sin 4\beta+\sin 2\beta\\
&+(2b\cos 2\alpha+6a+4b+4c+4e)\big]\Big]\Bigg].
\end{aligned}
$$

So if $D_3=b\sin 2\alpha-c\sin 2\beta+e\sin 2(\alpha-\beta)=0$,

$$
f_{13}(r,\alpha,R,\beta)=0\Rightarrow\begin{cases} c=0,\\ R=0,\\ \beta=\dfrac{k\pi}{2} \quad \text{with} \quad k\in\mathbb{Z}.\end{cases}
$$

and if $D_3\neq 0$ we get

$$
f_{13}(r,\alpha,R,\beta)=0\Rightarrow R=\sqrt{\frac{2bh\sin 2\alpha}{D_3}}.
$$

Subcase 2.1: $D_3=0$ and $c=0$. No information as in subcase 1.1. Hence, *in what follows in the rest of case 2 we assume that* $c\neq 0$.

Subcase 2.2: $D_3=0$ and $R=0$. Then $r=\sqrt{-2h}$, $h<0$ and $\rho=0$.

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = \;& \frac{1}{4}\big[h(3a+2b+b\cos 2\alpha)\big],\\
f_{14}(r,\alpha,R,\beta) = \;& \frac{1}{4}\big[h(3a+2c+c\cos 2\beta)\big].
\end{aligned}
$$

If $b\neq 0$ and $c\neq 0$, solving $f_{12}=f_{14}=0$ we get
$\alpha=\pm\dfrac{1}{2}\arccos-\dfrac{3a+2b}{b}$ and $\beta=\pm\dfrac{1}{2}\arccos-\dfrac{3a+2c}{c}$.
With the condition that

$$
h<0,\quad bc\neq 0,\quad \left|\frac{3a+2b}{b}\right|<1 \quad \text{and} \quad \left|\frac{3a+2c}{c}\right|<1, \tag{30}
$$

system (17) has four solutions $S^* = (r, \alpha, R, \beta)$ with $\rho = 0$ given by

$$(\sqrt{-2h}, \pm\frac{1}{2}\arccos -\frac{3a+2b}{b}, 0, \pm\frac{1}{2}\arccos -\frac{3a+2c}{c}), \tag{31}$$

which reduce to two solutions if either $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| < 1$ and $\left|\frac{3a+2c}{c}\right| = 1$ or $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| = 1$ and $\left|\frac{3a+2c}{c}\right| < 1$, and to one solution if $h < 0$, $bc \neq 0$, $\left|\frac{3a+2b}{b}\right| = 1$ and $\left|\frac{3a+2c}{c}\right| = 1$.

The Jacobian $J_{f_1(S^*)} = -\frac{9h^4}{32}(a+b)(3a+b)(a+c)(3a+c)$.

Assuming that $D_3 = 0$, $(a+b)(3a+b)(a+c)(3a+c) \neq 0$ and (30) hold. This supposition is true for the value $a = -(2/3), b = 1, c = 4/3, h = -1$. Then $J_{f_1(S^*)} \neq 0$ and the four solutions (31) of system (17) provide only one periodic solution of differential system (15) because when $R = 0$ and $\rho = 0$ the two solutions of both α and β provide the same initial conditions in (10).

If $b = 0$, $f_{12} = 3ah/4$=constant.

If $c = 0$, $f_{14} = 3ah/4$=constant.

Subcase 2.3: $D_3 = 0$ and $\beta = \frac{k\pi}{2}$. Due to the periodicity of the cosinus we study the cases $k = 0$ and $k = 2$, and the cases $k = 1$ and $k = 3$, together.

Subcase 2.3.1: Assume that either $k = 0$ or $k = 2$, i.e. either $\beta = 0$ or $\beta = \pi$. Then substituting β in r we obtain $r = \sqrt{R^2 - 2h}$. f_{12} and f_{14} become

$$\begin{aligned} f_{12}(r, \alpha, R, \beta) = &\ \frac{1}{8}\big[6ah + 4bh - (3a + 2b + 3c + 2e)R^2 \\ &+ (2bh - (b+e)R^2)\cos 2\alpha\big], \\ f_{14}(r, \alpha, R, \beta) = &\ \frac{3}{8}\big[2(a+c)h - (a + 2c + f)R^2\big]. \end{aligned}$$

If $a + 2c + f \neq 0$, solving $f_{14} = 0$ with respect to R we obtain $R = \sqrt{2h(a+c)/a + 2c + f}$. Substituting R in f_{12} and assuming that $bc - ae - ce + bf \neq 0$ we obtain solving $f_{12} = 0$ with respect to α, $\alpha = \pm\frac{1}{2}\arccos \Delta_3$, where $\Delta_3 = \frac{3c^2 - 2bc + 2ae + 2ce - 3af - 2bf}{bc - ae - ce + bf}$. Substituting R in r and in ρ we obtain $r = \sqrt{\frac{-2h(c+f)}{a + 2c + f}}$ and $\rho = 0$.

Supposing that

$$\begin{aligned} &a + 2c + f \neq 0, \quad (bc - ae - ce + bf) \neq 0, \\ &h(a + 2c + f)(c + f) < 0, \quad h(a + 2c + f)(a + c) > 0 \quad \text{and} \quad |\Delta_3| < 1. \end{aligned} \tag{32}$$

System (17) for $k=0$ and $k=2$ has four zeros solutions $S^*=(r,\alpha,R,\beta)$ with $\rho=0$ given by

$$\begin{array}{l}\left(\sqrt{\dfrac{-2h(c+f)}{a+2c+f}},\pm\dfrac{1}{2}\arccos\Delta_3,\sqrt{\dfrac{2h(a+c)}{a+2c+f}},0\right),\\ \left(\sqrt{\dfrac{-2h(c+f)}{(a+2c+f)}},\pm\dfrac{1}{2}\arccos\Delta_3,\sqrt{\dfrac{2h(a+c)}{a+2c+f}},\pi\right),\end{array}\tag{33}$$

which reduce to two solutions if $(bc-ae-ce+bf)\neq 0$, $a+2c+f\neq 0$, $|\Delta_3|=1$, $h(a+2c+f)(c+f)<0$ and $h(a+2c+f)(a+c)>0$.

The Jacobian is

$$\begin{array}{rl}J_{f_1(S^*)}= & \dfrac{-9ch^4}{16(a+2c+f)^3}(a+c)(c+f)\big(bc-c^2-ae-ce+af+bf\big)\\ & \big(bc-3c^2-ae-ce+3af+bf\big).\end{array}$$

In the case that $(bc-c^2-ae-ce+af+bf)\big(bc-3c^2-ae-ce+3af+bf\big)\neq 0$ and (32) hold, this supposition is not empty because the value $a=2,b=-3,c=-1,f=-1,e=0,h=-1$ satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (33) of system (17) provide only two periodic solutions of differential system (15) because since $\rho=0$ the two solutions of α provide the same initial conditions in (10).

If $a+2c+f=0$, $f_{14}=3h(a+c)/4$=constant.

If $a+2c+f\neq 0$ and $bc+bf-ae-ce=0$ we get $f_{12}=h(3af+2b(c+f)-3c^2-2ae-2ce)/4(a+2c+f)$=constant.

Subcase 2.3.2: Assume that either $k=1$ or $k=3$, i.e. either $\beta=\dfrac{\pi}{2}$ or $\beta=\dfrac{3\pi}{2}$. Then substituting β in r we get $r=\sqrt{R^2-2h}$. f_{12} and f_{14} become

$$\begin{array}{rl}f_{12}(r,\alpha,R,\beta)= & \dfrac{1}{8}\big[6ah+4bh-(3a+2b+c+2e)R^2\\ & +(2bh-bR^2+eR^2)\cos 2\alpha\big],\\ f_{14}(r,\alpha,R,\beta)= & \dfrac{1}{8}\big[2h(3a+c)-(3a+2c+3f)R^2\big].\end{array}$$

Solving $f_{14}=0$ we get if $3a+2c+3f\neq 0$, $R=\sqrt{\dfrac{2h(3a+c)}{3a+2c+3f}}$. Substituting R in f_{12} and solving $f_{12}=0$, we obtain if $bc+3ae+ce+3bf\neq 0$, $\alpha=\pm\dfrac{1}{2}\arccos\Delta_4$ where $\Delta_4=(-2bc+c^2+6ae+2ce-9af-6bf)/bc+3ae+ce+3bf$. Substituting R in r and in ρ we have $r=\sqrt{\dfrac{-2h(c+3f)}{3a+2c+3f}}$ and $\rho=0$.

Assuming that

$$\begin{array}{l}3a+2c+3f\neq 0,\quad bc+3ae+ce+3bf\neq 0,\\ |\Delta_4|<1,\quad h(3a+2c+3f)(c+3f)<0,\\ \text{and}\quad h(3a+2c+3f)(3a+c)>0.\end{array}\tag{34}$$

System (17) for $k = 1$ and $k = 3$ has four solutions $S^* = (r, \alpha, R, \beta)$ with $\rho = 0$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{-2h(c+3f)}{3a+2c+3f}}, \pm\frac{1}{2}\arccos\Delta_4, \sqrt{\frac{2h(3a+c)}{3a+2c+3f}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{-2h(c+3f)}{3a+2c+3f}}, \pm\frac{1}{2}\arccos\Delta_4, \sqrt{\frac{2h(3a+c)}{3a+2c+3f}}, \frac{3\pi}{2}\right),
\end{aligned} \tag{35}
$$

which reduce to two solutions if $3a+2c+3f \neq 0$, $bc+3ae+ce+3bf \neq 0$, $|\Delta_4| = 1$, $h(3a+2c+3f)(c+3f) < 0$ and $h(3a+2c+3f)(3a+c) > 0$.

The Jacobian is

$$
\begin{aligned}
J_{f_1(S^*)} = \ & \frac{ch^4(3a+c)(c+3f)}{(163a+2c+3f)^3}\big(-3bc+c^2+3ae+ce-9af-9bf\big)\\
& \big(-bc+c^2+9ae+3ce-9af-3bf\big).
\end{aligned}
$$

Conceding that $c\big(-3bc+c^2+3ae+ce-9af-9bf\big)\big(-bc+c^2+9ae+3ce-9af-3bf\big) \neq 0$ and (34) hold. This assumption is not empty because the value $a=1, b=-2, c=-1, f=-1, e=0, h=-1$ satisfy it. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (35) of system (17) provide only two periodic solutions of differential system (15) because since $\rho = 0$ the two solutions of α provide the same initial conditions in (10).

If $3a+2c+3f = 0$, we have $f_{14} = h(3a+c)/4$=constant.

If $3a+2c+3f \neq 0$ and $bc+3ae+ce+3bf = 0$, then $f_{12} = h(9af+2b(c+3f)-c^2-6ae-2ce)/4(3a+2c+3f)$=constant.

Subcase 2.4: $D_3 \neq 0$ and $R = \sqrt{\dfrac{2bh\sin 2\alpha}{D_3}}$. Then $r = \sqrt{\dfrac{2eh\sin 2(\beta-\alpha)}{D_3}}$. f_{12}, f_{14} become

$$
\begin{aligned}
f_{12}(r,\alpha,R,\beta) = \ & \frac{h}{4D_3}\big[(3ae-bc+2be)\sin 2(\alpha-\beta)+(2bc-be\\
& +3cd)\sin 2\beta - 2b(c+e)\sin 2\alpha\big],\\
f_{14}(r,\alpha,R,\beta) = \ & \frac{h}{4D_3}\big[(-2bc+ce-3bf)\sin 2\alpha + (-bc+3ae+2ce)\\
& \sin 2(\alpha-\beta) + 2c(b+e)\sin 2\beta\big].
\end{aligned}
$$

To calculate the zeros of these two last functions we need their numerators $h\big[(3ae-bc+2be)\sin 2(\alpha-\beta)+(2bc-be+3cd)\sin 2\beta - 2b(c+e)\sin 2\alpha\big]$, $h\big[(-2bc+ce-3bf)\sin 2\alpha + (-bc+3ae+2ce)\sin 2(\alpha-\beta)+2c(b+e)\sin 2\beta\big]$. Expanding the trigonometrical terms of these numerators and using the notation $\sin\alpha = s$; $\cos\alpha = \pm\sqrt{1-s^2}$; $\sin\beta = S$; $\cos\beta = \pm\sqrt{1-S^2}$ we obtain using the sign $+$ for $\cos\alpha$ and $\cos\beta$

$$
\begin{aligned}
P_{12}(s,S) = \ & 2hs\sqrt{1-s^2}\left(-6aeS^2+3ae+2bcS^2-3bc-4beS^2\right)\\
& -2hS\sqrt{1-S^2}\big(-6aes^2+3ae+2bcs^2-3bc-4bes^2\\
& +3be-3cd\big),\\
P_{14}(s,S) = \ & 2hs\sqrt{1-s^2}\big(-6aeS^2+3ae+2bcS^2-3bc-3bf\\
& -4ceS^2+3ce\big)\\
& -2hS\sqrt{1-S^2}\left(-6aes^2+3ae+2bcs^2-3bc-4ces^2\right).
\end{aligned}
$$

Note that the other three subcases provide, taking into account the different signs, the same zeros than the system $P_{12}(s,S) = P_{14}(s,S) = 0$. This last system is equivalent to the system $Q_{12}(s,S) = Q_{14}(s,S) = 0$ where

$$
\begin{aligned}
Q_{12}(s,S) &= 4h^2S^2\left(1-S^2\right)\big(-6aes^2+3ae+2bcs^2 \\
&\quad -3bc-4bes^2+3be-3cd\big)^2-4h^2s^2\left(1-s^2\right) \\
&\quad \big(-6aeS^2+3ae+2bcS^2-3bc-4beS^2\big)^2, \\
Q_{14}(s,S) &= 4h^2S^2\left(1-S^2\right)\big(-6aes^2+3ae+2bcs^2 \\
&\quad -3bc-4ces^2\big)^2-4h^2s^2\left(1-s^2\right)\big(-6aeS^2 \\
&\quad +3ae+2bcS^2-3bc-3bf-4ceS^2+3ce\big)^2.
\end{aligned} \tag{36}
$$

Calculating the resultant of Q_{12} and Q_{14} with respect to s and S, we obtain

$$
\begin{aligned}
R_{12}(S) &= 47775744h^{16}(-1+S)^4S^8(1+S)^4T^2(S)U^2(S), \\
R_{14}(s) &= 47775744h^{16}(-1+s)^4s^8(1+s)^4V^2(s)W^2(s),
\end{aligned} \tag{37}
$$

with T(S) and V(s) two polynomials of the form AS^2+B and Cs^2+D respectively with A, B, C, D constants, and U(S) and W(s) two polynomials of the form ES^4+FS^2+G and Hs^4+Is^2+J respectively with E, F, G, H, I, J constants. Solving (37) we obtain 81 pairs (s,S). Only 9 of these pairs are solutions of (36). When we calculate (α,β) corresponding to an (s,S) solution we find the zeros $S^* = (r^*,\rho^*,\alpha^*,R^*,\beta^*)$ of (17) given by

$$S^*_{1,\pm,\pm} = \left(\sqrt{\frac{2eh}{c-e}},\sqrt{\frac{2ch}{c-e}},\pm\frac{\pi}{2},0,\pm\frac{\pi}{2}\right);$$

$$S^*_{2,\pm} = \left(\sqrt{\frac{2eh}{c-e}},\sqrt{\frac{2ch}{c-e}},\pm\frac{\pi}{2},0,0\right);$$

$$S^*_{3,\pm} = \left(\sqrt{\frac{-2eh}{c-e}},\sqrt{\frac{2h(c-2e)}{c-e}},0,0,\pm\frac{\pi}{2}\right);$$

$$S^*_{4} = \left(\sqrt{\frac{-2eh}{c-e}},\sqrt{\frac{2h(c-2e)}{c-e}},0,0,0\right).$$

The values of r and R are not well defined in these solutions. So the averaging theory in subcase 2.4 does not give information.

Case 3: $b\sin 2\alpha = 0$. Then $b = 0$ or $\sin 2\alpha = 0$.

Subcase 3.1: $b = 0$. Then $f_{11} = -crR^2\sin(2\beta)/8$. Solving $f_{11} = 0$ we obtain the following four subcases: $c = 0$, $r = 0$ (studied in case 1), $R = 0$, $\beta = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$.

Subcase 3.1.1: $c = 0$. No information as in subcase 1.1. Hence *in what follows in subcase* 3.1 *we assume that* $c \neq 0$.

Subcase 3.1.2: $R = 0$. This subcase does not give results because f_{13} and f_{11} will be zero.

Subcase 3.1.3 $\beta = \dfrac{m\pi}{2}$ with $m \in \mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $m = 0$ and $m = 2$, and the subcases $m = 1$ and $m = 3$ together.

Subcase 3.1.3.1: Assume that either $\beta = 0$ or $\beta = \pi$. So $f_{13} = \dfrac{eR}{8}(2h + r^2 - R^2)\sin 2\alpha$.

$$f_{13}(r,\alpha,R,\beta) = 0 \Rightarrow \begin{cases} e = 0, \\ R = 0, \\ R = \sqrt{2h + r^2}, \\ \alpha = \dfrac{n\pi}{2} \quad \text{with} \quad n \in \mathbb{Z}. \end{cases}$$

Subcase 3.1.3.1.1: $e = 0$. No information as in subcase 1.1. Hence, *in what follows in subcase* 3.1.4 *we assume that* $e \neq 0$.

Subcase 3.1.3.1.2: $R = 0$. Then we have

$$\begin{aligned} f_{12} =& \quad -\frac{3}{8}[2dh + (a+d)r^2], \\ f_{14} =& \quad -\frac{1}{8}[(3a + 3c + 2e + e\cos 2\alpha)r^2 + 2eh(2 + \cos 2\alpha)]. \end{aligned}$$

If $a + d \neq 0$, $f_{12} = 0 \Rightarrow r = \sqrt{\dfrac{-2dh}{a+d}}$. So $\rho = \sqrt{\dfrac{2ah}{a+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ if $ae \neq 0$, we get $\alpha = \pm\dfrac{1}{2}\arccos\dfrac{3ad + 3cd - 2ae}{ae}$.

In the case that $\left|\dfrac{3ad + 3cd - 2ae}{ae}\right| < 1$, $ae(a+d) \neq 0$, $hd(a+d) < 0$ and $ah(a+d) > 0$, system (17) for $m = 0$ and $m = 2$ has four zeros $S^* = (r,\alpha,R,\beta)$ with $\rho = \sqrt{\dfrac{2ah}{a+d}}$ given by

$$\left(\sqrt{-\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{3ad + 3cd - 2ae}{ae}\right), 0, 0\right),$$
$$\left(\sqrt{-\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\left(\frac{3ad + 3cd - 2ae}{ae}\right), 0, \pi\right),$$

which reduce to two zeros when $ae(a+d) \neq 0$, $hd(a+d) < 0$, $ah(a+d) > 0$ and $\left|\dfrac{3ad + 3cd - 2ae}{ae}\right| = 1$. But when the Jacobian is evaluated on these solutions it becomes zero so the averaging theory does not give information in this subcase.

If $a + d = 0$, we have $f_{12} = -3dh/4$=constant.

If $a + d \neq 0$ and $ae = 0$, $f_{14} = \dfrac{3hd(a+c)}{4(a+d)}$=constant.

Subcase 3.1.3.1.3: $R = \sqrt{2h + r^2}$. Studied in the subcase 2.4.1.

Subcase 3.1.3.1.4: $\alpha = \dfrac{n\pi}{2}$. Due to the periodicity of the sinus we study the cases $n = 0$ and $n = 2$, and the cases $n = 1$ and $n = 3$ together.

Subcase 3.1.3.1.4.1: Assume that either $n = 0$ or $n = 2$, i.e. either $\alpha = 0$ or $\alpha = \pi$.

$$\begin{aligned} f_{12} =& \quad -\frac{1}{8}\left[6dh + 3(a+d)r^2 + 3(c - d + e)R^2\right], \\ f_{14} =& \quad -\frac{1}{8}\left[6eh + 3(a + c + e)r^2 - 3(e - c - f)R^2\right]. \end{aligned}$$

If $a+c+e=0$ and $(e-c-f)\neq 0$, solving $f_{14}=0$ we get $R=\sqrt{\dfrac{2eh}{e-c-f}}$. Substituting R in f_{12} and solving $f_{12}=0$ if $a+d\neq 0$, we have $r=\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}}$ and $\rho=\sqrt{\dfrac{-2h(ce+e^2+ac+af))}{(a+d)(e-c-f)}}$.

Supposing that

$$\begin{array}{l} a+c+e=0, \quad e-c-f\neq 0, \quad eh(e-c-f)>0 \\ h(cd+df-ce-e^2)(a+d)(e-c-f)>0 \quad \text{and} \\ h(ce+e^2+ac+af)(a+d)(e-c-f)<0. \end{array} \tag{38}$$

System (17) for $n=0$ and $n=2$ has the following four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{-2h(ce+e^2+ac+af)}{(a+d)(e-c-f)}}$ given by

$$\begin{array}{c} \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},0,\sqrt{\dfrac{2eh}{e-c-f}},0\right), \\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},\pi,\sqrt{\dfrac{2eh}{e-c-f}},0\right), \\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},0,\sqrt{\dfrac{2eh}{e-c-f}},\pi\right), \\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{(a+d)(e-c-f)}},\pi,\sqrt{\dfrac{2eh}{e-c-f}},\pi\right). \end{array} \tag{39}$$

The Jacobian

$$\begin{aligned} J_{f_1(s*)} = {} & \frac{9ce^3h^4}{16(a+d)^2(e-c-f)^4}(ce+e^2+ac+af)(cd+df-ce \\ & -e^2)\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big). \end{aligned}$$

With the condition $ce\big(ad-c^2+2cd-2ae-2ce-e^2+af+df\big)\neq 0$ and (38) we have $J_{f_1(S^*)}\neq 0$. The condition is not empty because the value $a=2, c=-1, d=-3, f=-1, e=-1, h=-1$ satisfy it. Therefore the four zeros (39) of (17) provide four periodic solutions of (15).

If $a+c+e=0$ and $(e-c-f)=0$, $f_{14}=-3eh/4$.

If $a+c+e=0$, $(e-c-f)\neq 0$ and $a+d=0$, $f_{12}=3h(ce+e^2-dc-df)/(4(e-c-f))$=constant.

If $a+c+e\neq 0$ and $(e-c-f)=0$, solving $f_{14}=0$ we get $r=\sqrt{\dfrac{-2eh}{a+c+e}}$. Substituting r in f_{12} and solving $f_{12}=0$ if $c-d+e\neq 0$, we have $R=\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}}$ and $\rho=\sqrt{\dfrac{2hc}{c-d+e}}$.

Assuming that

$$\begin{array}{l} a+c+e\neq 0, \quad c-e+f=0, \quad c-d+e\neq 0, \\ eh(a+c+e)<0, \quad hc(c-d+e)>0 \quad \text{and} \\ h(ae-ad-cd)(a+c+e)(c-d+e)>0. \end{array} \tag{40}$$

System (17) for $n=0$ and $n=2$ has the following four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2hc}{c-d+e}}$ given by

$$(41)\qquad \begin{array}{l} \left(\sqrt{\dfrac{-2eh}{a+c+e}},0,\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}},0\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}},\pi,\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}},0\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}},0,\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}},\pi\right),\\ \left(\sqrt{\dfrac{-2eh}{a+c+e}},\pi,\sqrt{\dfrac{2h(ae-ad-cd)}{(a+c+e)(c-d+e)}},\pi\right). \end{array}$$

The Jacobian

$$\begin{aligned} J_{f_1(s*)} = & -\frac{9c^2e^2h^4}{16(a+c+e)^3(c-d+e)^3}(ad+cd-ae)^2(c^2-ad-2cd \\ & +2ae+2ce+e^2-af-df). \end{aligned}$$

With the condition $ce\big(c^2-ad-2cd+2ae+2ce+e^2-af-df\big)\neq 0$ and (40), we have $J_{f_1(S^*)}\neq 0$. The set of conditions is not empty because it is satisfied for the value $a=9/4, c=-1, d=-4, e=-2, f=-1, h=-1$. Then the four zeros (41) of (17) provide four periodic solutions of (15).

If $a+c+e\neq 0$, $(e-c-f)=0$ and $c-d+e=0$, $f_{12}=-3h(ad+cd-ae)/(4(a+c+e))$=constant.

If $a+c+e\neq 0$ and $e-c-f\neq 0$. Then solving $f_{14}=0$ we get $r=\sqrt{\dfrac{(e-c-f)R^2-2he}{a+c+e}}$. Substituting r in f_{12} and solving $f_{12}=0$ we have, if
$\Sigma=c^2-ad-2cd+2ae+2ce+e^2-af-df\neq 0$,
$R=\sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}}$ then $r=\sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}}$ and
$\rho=\sqrt{\dfrac{2h(ce+c^2+ae-af)}{\Sigma}}$.
Wherever

$$(42)\qquad \begin{array}{l} \Sigma=c^2-ad-2cd+2ae+2ce+e^2-af-df\neq 0,\\ a+c+e\neq 0,\quad e-c-f\neq 0,\quad h(cd+df-ce-e^2)\Sigma>0,\\ h(ae-ad-cd)\Sigma>0 \quad\text{and}\quad h(ce+c^2+ae-af)\Sigma>0, \end{array}$$

system (17) for $n=0$ and $n=2$ has four zeros
$S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(ce+c^2+ae-af)}{\Sigma}}$ given by

$$\begin{array}{c}\left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}},0,\sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}},0\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}},\pi,\sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}},0\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}},0,\sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}},\pi\right),\\ \left(\sqrt{\dfrac{2h(cd+df-ce-e^2)}{\Sigma}},\pi,\sqrt{\dfrac{2h(ae-ad-cd)}{\Sigma}},\pi\right).\end{array} \tag{43}$$

The Jacobian is

$$J_{f_1(s*)}=\frac{9ceh^4}{16\Sigma^3}(cd+df-ce-e^2)(ce+c^2+ae-af)(ae-ad-cd)^2.$$

Assuming that $ce\neq 0$ and (42) hold. This supposition is not empty because the value $a=5,c=-2,d=-2,f=-2,e=-1,h=1$, satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (43) of (17) provide four periodic solutions of (15).

If $a+c+e\neq 0$, $e-c-f\neq 0$ and $\Sigma=0$ we have $f_{12}=-3h(ad+cd-ae)/(4(a+c+e))$=constant.

Subcase 3.1.3.1.4.2: Assume that either $n=1$ or $n=3$, i.e. either $\alpha=\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. Then

$$\begin{aligned} f_{12}= &\ -\frac{1}{8}\left[6dh+3(a+d)r^2+(3c-3d+e)R^2\right],\\ f_{14}= &\ -\frac{1}{8}\left[2eh+(3a+3c+e)r^2+(3c+3f-e)R^2\right]. \end{aligned}$$

If $3c-3d+e\neq 0$ and $a+d=0$, solving $f_{12}=0$ we get
$R=\sqrt{-\dfrac{6dh}{3c-3d+e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to R if $3a+3c+e\neq 0$, we obtain $r=\sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{(3c-3d+e)(3a+3c+e)}}$,
$\rho=\sqrt{\dfrac{6h(3ac+ae+3c^2+3cd+ce+3df)}{(3c-3d+e)(3a+3c+e)}}$.

Whenever

$$\begin{array}{l} 3c-3d+e\neq 0,\quad a+d=0,\quad 3a+3c+e\neq 0,\\ dh(3c-3d+e)<0,\quad h(9cd-3ce-e^2+9df)(3c-3d+e)\\ (3a+3c+e)>0\quad \text{and}\quad h(3ac+ae+3c^2+3cd+ce+3df)\\ (3c-3d+e)(3a+3c+e)>0, \end{array} \tag{44}$$

system (17) for $n=1$ and $n=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with

$\rho = \sqrt{\dfrac{6h(3ac+ae+3c^2+3cd+ce+3df)}{(3c-3d+e)(3a+3c+e)}}$ given by

$$
\begin{array}{l}
\left(\sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{(3c-3d+e)(3a+3c+e)}}, \dfrac{\pi}{2}, \sqrt{-\dfrac{6dh}{(3c-3d+e)}}, 0\right),\\
\left(\sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{(3c-3d+e)(3a+3c+e)}}, \dfrac{3\pi}{2}, \sqrt{-\dfrac{6dh}{(3c-3d+e)}}, 0\right),\\
\left(\sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{(3c-3d+e)(3a+3c+e)}}, \dfrac{\pi}{2}, \sqrt{-\dfrac{6dh}{(3c-3d+e)}}, \pi\right),\\
\left(\sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{(3c-3d+e)(3a+3c+e)}}, \dfrac{3\pi}{2}, \sqrt{-\dfrac{6dh}{(3c-3d+e)}}, \pi\right).
\end{array}
\tag{45}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = & -\frac{27cd^2eh^4}{16(3a+3c+e)^2(3c-3d+e)^4}(9cd-3ce-e^2+9df)(3ac\\
& +ae+3c^2+3cd+ce+3df)(9c^2-9ad-18cd+6ae+6ce\\
& +e^2-9af-9df).
\end{aligned}
$$

In case that $cde(9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df) \neq 0$ and (44) hold, the set of these conditions is not empty because the value $a=2, c=-1, d=-2, f=0, e=-1, h=1$, satisfy it. Therefore we have $J_{f_1(S^*)} \neq 0$ and the four zeros (45) of (17) provide four periodic solutions of differential system (15).

If $3c-3d+e=0$ and $a+d=0$ we get $f_{12} = -3dh/4$=constant.

If $3c-3d+e \neq 0$, $a+d=0$ and $3a+3c+e=0$ we have $f_{14} = h(9cd-3ce-e^2+9df)/(4(3c-3d+e))$=constant.

If $3c-3d+e=0$ and $a+d \neq 0$, $f_{12}=0 \Rightarrow r = \sqrt{\dfrac{-2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R if $3c-e+3f \neq 0$ we obtain $R = \sqrt{\dfrac{2h(3ad+3cd-ae)}{(a+d)(3c-e+3f)}}$, $\rho = \sqrt{\dfrac{6h(ac-ad+af-cd)}{(a+d)(3c-e+3f)}}$.

Assuming that

$$
\begin{array}{l}
3c-3d+e=0, \quad a+d \neq 0, \quad 3c-e+3f \neq 0,\\
dh(a+d)<0, \quad h(3ad+3cd-ae)(a+d)(3c-e+3f)>0\\
\text{and} \quad h(ac-ad+af-cd)(a+d)(3c-e+3f)>0.
\end{array}
\tag{46}
$$

System (17) for $n=1$ and $n=3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with

$\rho = \sqrt{\dfrac{6h(ac-ad+af-cd)}{(a+d)(3c-e+3f)}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{-2dh}{a+d}}, \frac{\pi}{2}, \sqrt{\frac{2h(3ad+3cd-ae)}{(a+d)(3c-e+3f)}}, 0\right),\\
&\left(\sqrt{\frac{-2dh}{a+d}}, \frac{3\pi}{2}, \sqrt{\frac{2h(3ad+3cd-ae)}{(a+d)(3c-e+3f)}}, 0\right),\\
&\left(\sqrt{\frac{-2dh}{a+d}}, \frac{\pi}{2}, \sqrt{\frac{2h(3ad+3cd-ae)}{(a+d)(3c-e+3f)}}, \pi\right),\\
&\left(\sqrt{\frac{-2dh}{a+d}}, \frac{3\pi}{2}, \sqrt{\frac{2h(3ad+3cd-ae)}{(a+d)(3c-e+3f)}}, \pi\right).
\end{aligned}
\tag{47}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = &\ \frac{3cdeh^4}{16(3c-e+3f)^3(a+d)^4}(3ad+3cd-ae)^2(ac-ad+af\\
&-cd)(9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df).
\end{aligned}
$$

With the condition that $cde(9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df) \neq 0$ and (46) hold, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (47) of (17) provide four periodic solutions of differential system (15). The set of conditions is not empty because it is satisfied for the value $a = 2, c = -2/3, d = -1, f = -1, e = -1, h = 1$.

If $a+d \neq 0$, $3c-3d+e = 0$ and $3c-e+3f = 0$ we get $f_{14} = (3ad+3cd-ae)h/(4(a+d))$=constant.

If $a+d \neq 0$ and $3c-3d+e \neq 0$, we have $r = \sqrt{-\dfrac{6dh+(3c-3d+e)R^2}{3(a+d)}}$. Substituting r in f_{14} and solving $f_{14} = 0$ if

$$
\Sigma_1 = 9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df \neq 0,
$$

we have $R = \sqrt{-\dfrac{6h(3ad+3cd-ae)}{\Sigma_1}}$, $r = \sqrt{\dfrac{2h(9cd-3ce-e^2+9df)}{\Sigma_1}}$ and $\rho = \sqrt{\dfrac{6h(3c^2+ce+ae-3af)}{\Sigma_1}}$.

Conceding that

$$
\begin{gathered}
\Sigma_1 = 9c^2-9ad-18cd+6ae+6ce+e^2-9af-9df \neq 0,\\
a+d \neq 0, \quad 3c-3d+e \neq 0, \quad h(9cd-3ce-e^2+9df)\Sigma_1 > 0,\\
h(3c^2+ce+ae-3af)\Sigma_1 > 0 \quad \text{and} \quad h(3ad+3cd-ae)\Sigma_1 < 0.
\end{gathered}
\tag{48}
$$

System (17) for $n=1$ and $n=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{6h(3c^2+ce+ae-3af)/\Sigma_1}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{2h(9cd-3ce-e^2+9df)}{\Sigma_1}},\frac{\pi}{2},\sqrt{\frac{-6h(3ad+3cd-ae)}{\Sigma_1}},0\right),\\
&\left(\sqrt{\frac{2h(9cd-3ce-e^2+9df)}{\Sigma_1}},\frac{3\pi}{2},\sqrt{\frac{-6h(3ad+3cd-ae)}{\Sigma_1}},0\right),\\
&\left(\sqrt{\frac{2h(9cd-3ce-e^2+9df)}{\Sigma_1}},\frac{\pi}{2},\sqrt{\frac{-6h(3ad+3cd-ae)}{\Sigma_1}},\pi\right),\\
&\left(\sqrt{\frac{2h(9cd-3ce-e^2+9df)}{\Sigma_1}},\frac{3\pi}{2},\sqrt{\frac{-6h(3ad+3cd-ae)}{\Sigma_1}},\pi\right).
\end{aligned}
\tag{49}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)}= & -\frac{27ceh^4}{16{\Sigma_1}^3}(3ad+3cd-ae)^2(3c^2+ae+ce-3af)(9cd-3ce\\
& -e^2+9df).
\end{aligned}
$$

Supposing $ce\neq 0$ and (48) hold. This assumption is not empty because the value $a=1/2, c=-1, d=-1, f=3, e=-1, h=-1$ satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (49) of system (17) provide four periodic solutions of differential system (15).

If $a+d\neq 0$, $3c-3d+e\neq 0$ and $\Sigma_1=0$ we get $f_{14}=h(3ad+3cd-ae)/(4(a+d))$=constant.

Subcase 3.1.3.2: Assume that either $\beta=\dfrac{\pi}{2}$ or $\beta=\dfrac{3\pi}{2}$. So if $f_{13}=\dfrac{eR}{8}(R^2-2h-r^2)\sin 2\alpha=0$, then consequently one of the following four subcases holds: $e=0$, $R=0$, $R=\sqrt{2h+r^2}$ and $\alpha=l\pi/2$ with $l\in\mathbb{Z}$.

Subcase 3.1.3.2.1: $e=0$. No information as in subcase 1.1. So *in what follows in this subcase we assume that* $e\neq 0$.

Subcase 3.1.3.2.2: $R=0$. Then

$$
\begin{aligned}
f_{12}= & -\frac{3}{8}[2dh+(a+d)r^2],\\
f_{14}= & -\frac{1}{8}[4eh+(3a+c+2e)r^2-e(2h+r^2)\cos 2\alpha].
\end{aligned}
$$

If $a+d\neq 0$ solving $f_{12}=0$, we obtain $r=\sqrt{-2dh/(a+d)}$ and $\rho=\sqrt{2ah/(a+d)}$. Substituting r in f_{14} and solving $f_{14}=0$ if $ae\neq 0$, we get $\alpha=\pm\dfrac{1}{2}\arccos\dfrac{2ae-3ad-cd}{ae}$. Therefore when $hd(a+d)<0$, $ah(a+d)>0$ and $\left|\dfrac{2ae-3ad-cd}{ae}\right|<1$, system (17) has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{2ah/(a+d)}$ given by $\left(\sqrt{-\dfrac{2dh}{a+d}},\pm\dfrac{1}{2}\arccos\dfrac{2ae-3ad-cd}{ae},0,\dfrac{\pi}{2}\right)$,

$$\left(\sqrt{-\frac{2dh}{a+d}}, \pm\frac{1}{2}\arccos\frac{2ae-3ad-cd}{ae}, 0, \frac{3\pi}{2}\right),$$

which reduces to two zeros if $hd(a+d) < 0$, $ah(a+d) > 0$ and $\left|\frac{2ae-3ad-cd}{ae}\right| = 1$. Evaluating the Jacobian on these zeros we get zero so the averaging theory does not give information in this subcase.

If $a+d=0$, we have $f_{12} = -3dh/4$=constant.

If $a+d \neq 0$ and $ae = 0$, we get $f_{14} = hd(3a+c)/(4(a+d))$=constant.

Subcase 3.1.3.2.3: $R = \sqrt{2h+r^2}$. Studied in the subcase 2.4.2.

Subcase 3.1.3.2.4: $\alpha = \frac{l\pi}{2}$ with $l \in \mathbb{Z}$. Due to the periodicity of the sinus we study the cases $l=0$ and $l=2$, and the cases $l=1$ and $l=3$ together.

Subcase 3.1.3.2.4.1: Assume that either $l=0$ or $l=2$, i.e. either $\alpha=0$ or $\alpha=\pi$. Then

$$\begin{aligned} f_{12} = &\quad -\frac{1}{8}\left[6dh + 3(a+d)r^2 + (c-3d+e)R^2\right], \\ f_{14} = &\quad -\frac{1}{8}\left[2he + (3a+c+e)r^2 + (c-e+3f)R^2\right]. \end{aligned}$$

If $3a+c+e=0$ and $c-e+3f \neq 0$, solving $f_{14}=0$, we obtain $R = \sqrt{-\frac{2he}{c-e+3f}}$. Substituting R in f_{12} and solving $f_{12}=0$ with respect to r if $a+d \neq 0$ we get $r = \sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}$ and $\rho = \sqrt{\frac{2h(ce+e^2+3ac+9af)}{3(a+d)(c-e+3f)}}$.

Supposing that

$$\begin{aligned} &3a+c+e=0, \quad c-e+3f \neq 0, \quad he(c-e+3f) < 0, \\ &h(a+d)(c-e+3f)(ce+e^2-9df-3cd) > 0, \quad h(a+d) \\ &(c-e+3f)(ce+e^2+3ac+9af) > 0 \quad \text{and} \quad a+d \neq 0. \end{aligned} \tag{50}$$

System (17) for $l=0$ and $l=2$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\frac{2h(ce+e^2+3ac+9af)}{3(a+d)(c-e+3f)}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, 0, \sqrt{\frac{-2he}{c-e+3f}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, \pi, \sqrt{\frac{-2he}{c-e+3f}}, \frac{\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, 0, \sqrt{\frac{-2he}{c-e+3f}}, \frac{3\pi}{2}\right), \\ &\left(\sqrt{\frac{2h(ce+e^2-9df-3cd)}{3(a+d)(c-e+3f)}}, \pi, \sqrt{\frac{-2he}{c-e+3f}}, \frac{3\pi}{2}\right). \end{aligned} \tag{51}$$

The Jacobian is

$$J_{f_1(s*)} = \frac{ce^3h^4}{144(a+d)^2(c-e+3f)^4}(3ac+ce+e^2+9af)(3cd-ce -e^2+9df)\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big).$$

Conceding that $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big)\neq 0$ and (50) hold. This assumption is not empty because it is satisfied for the value $a=2/3, d=-1, c=-1, f=-1, e=-1, h=-1$. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (51) of system (17) provide four periodic solutions of differential system (15).

If $3a+c+e=0$ and $c-e+3f=0$ we have $f_{14}=-eh/4$=constant.

If $3a+c+e=0$, $c-e+3f\neq 0$ and $a+d=0$, we get $f_{12}=h(2ec+2e^2-6dc-18df)/(8(c-e+3f))$=constant.

If $3a+c+e\neq 0$ and $c-e+3f=0$, solving $f_{14}=0$, we obtain $r=\sqrt{-2he/(3a+c+e)}$. Substituting r in f_{12} and solving $f_{12}=0$ with respect to R if $3d-c-e\neq 0$ we get $R=\sqrt{\dfrac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}$ and $\rho=\sqrt{\dfrac{-2ch}{3d-c-e}}$.

Supposing that

$$\begin{gathered} 3a+c+e\neq 0, \quad c-e+3f=0, \quad 3d-c-e\neq 0,\\ he(3a+c+e)<0, \quad hc(3d-c-e)<0. \quad \text{and}\\ h(3ad+cd-ae)(3d-c-e)(3a+c+e)>0. \end{gathered} \tag{52}$$

System (17) for $l=0$ and $l=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{-2ch}{3d-c-e}}$ given by

$$\begin{gathered} \left(\sqrt{\frac{-2he}{3a+c+e}}, 0, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{\pi}{2}\right),\\ \left(\sqrt{\frac{-2he}{3a+c+e}}, \pi, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{\pi}{2}\right),\\ \left(\sqrt{\frac{-2he}{3a+c+e}}, 0, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{3\pi}{2}\right),\\ \left(\sqrt{\frac{-2he}{3a+c+e}}, \pi, \sqrt{\frac{6h(3ad+cd-ae)}{(3d-c-e)(3a+c+e)}}, \frac{3\pi}{2}\right). \end{gathered} \tag{53}$$

The Jacobian is

$$J_{f_1(s*)} = \frac{9c^2e^2h^4(3ad+cd-ae)^2(c^2-9ad-6cd+6ae+2ce+e^2-9af-9df)}{16(3d-c-e)^3(3a+c+e)^3}.$$

Supposing that $ce\big(9ad-c^2+6cd-6ae-2ce-e^2+9af+9df\big)\neq 0$ and (52) hold. This assumption is not empty because it is satisfied for the value $a=3/2, d=-2, c=-1, f=-1, e=-4, h=-1$. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (53) of system (17) provide four periodic solutions of differential system (15).

If $3a+c+e\neq 0$, $c-e+3f=0$ and $3d-c-e=0$, we get $f_{12}=-3h(3ad+cd-ae)/(4(3a+c+e))$=constant.

If $3a+c+e\neq 0$ and $c-e+3f\neq 0$, solving $f_{14}=0$ we get $R=\sqrt{-\dfrac{2he+(3a+c+e)r^2}{(c-e+3f)}}$. Substituting R in f_{12} and solving $f_{12}=0$ with respect to r if

$\Sigma_2=c^2-9ad-6cd+6ae+2ce+e^2-9af-9df\neq 0$

we get $r=\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}}$, $R=\sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}}$ and $\rho=\sqrt{\dfrac{2h(c^2+ce+3ae-9af)}{\Sigma_2}}$.

Supposing that

$$\begin{array}{l}\Sigma_2=c^2-9ad-6cd+6ae+2ce+e^2-9af-9df\neq 0,\\ h(ae-3ad-cd)\Sigma_2>0,\quad 3a+c+e\neq 0,\quad c-e+3f\neq 0\\ h(3cd-ce-e^2+9df)\Sigma_2>0\quad\text{and}\quad h(c^2+ce+3ae-9af)\Sigma_2>0.\end{array}\tag{54}$$

System (17) for $l=0$ and $l=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(c^2+ce+3ae-9af)}{\Sigma_2}}$ given by

$$\begin{array}{l}\left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}},0,\sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}},\pi,\sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}},\dfrac{\pi}{2}\right),\\ \left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}},0,\sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}},\dfrac{3\pi}{2}\right),\\ \left(\sqrt{\dfrac{2h(3cd-ce-e^2+9df)}{\Sigma_2}},\pi,\sqrt{\dfrac{6h(ae-3ad-cd)}{\Sigma_2}},\dfrac{3\pi}{2}\right).\end{array}\tag{55}$$

Its Jacobian is

$$\begin{aligned}J_{f_1(s*)}= & \frac{9ceh^4}{16\Sigma_2^3}(ae-3ad-cd)^2(c^2+ce+3ae-9af)(3cd-ce-e^2\\ & +9df).\end{aligned}$$

Assuming that $ce\neq 0$ and (54) hold, this supposition is not empty because the value $a=-1, d=0, c=-1, f=-1, e=5/2, h=1$ satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (55) of system (17) provide four periodic solutions of differential system (15).

If $3a+c+e\neq 0$, $c-e+3f\neq 0$ and $\Sigma_2=0$ we have $f_{12}=-3h(3ad+cd-ae)/(4(3a+c+e))$=constant.

Subcase 3.1.3.2.4.2: Assume that either $l=1$ or $l=3$. Then

$$\begin{aligned}f_{12}= & -\frac{1}{8}\big[6dh+3(a+d)r^2+(c-3d+3e)R^2\big],\\ f_{14}= & -\frac{1}{8}\big[(3a+c+3e)r^2+(c-3e+3f)R^2+6eh\big].\end{aligned}$$

If $a+d=0$ and $c-3d+3e\neq 0$, solving $f_{12}=0$ we get $R=\sqrt{\dfrac{-6dh}{c-3d+3e}}$. Substituting R in f_{14} and solving $f_{14}=0$ we obtain, if $3a+c+3e\neq 0$, $r=\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}}$ and $\rho=\sqrt{\dfrac{2h(3ac+9ae+c^2+3cd+3ce+9df)}{(c-3d+3e)(3a+c+3e)}}$.

Conceding that

$$
\begin{aligned}
&a+d=0,\quad c-3d+3e\neq 0,\quad 3a+c+3e\neq 0,\\
&dh(c-3d+3e)<0,\quad h(cd-ce-3e^2+3df)(c-3d+3e)\\
&(3a+c+3e)>0\quad\text{and}\quad h(3ac+9ae+c^2+3cd+3ce+9df)\\
&(c-3d+3e)(3a+c+3e)>0.
\end{aligned}
\tag{56}
$$

System (17) for $l=1$ and $l=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(3ac+9ae+c^2+3cd+3ce+9df)}{(c-3d+3e)(3a+c+3e)}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}},\frac{\pi}{2},\sqrt{\frac{-6dh}{(c-3d+3e)}},\frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}},\frac{3\pi}{2},\sqrt{\frac{-6dh}{(c-3d+3e)}},\frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}},\frac{\pi}{2},\sqrt{\frac{-6dh}{(c-3d+3e)}},\frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{6h(cd-ce-3e^2+3df)}{(c-3d+3e)(3a+c+3e)}},\frac{3\pi}{2},\sqrt{\frac{-6dh}{(c-3d+3e)}},\frac{3\pi}{2}\right).
\end{aligned}
\tag{57}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \;& \frac{9dh^4}{16(3a+c+3e)^3(c-3d+3e)^4}(cd-ce-3e^2+3df)(3ac\\
&+c^2+3cd+9ae+3ce+9df)(-b^2+2bc-c^2+9ad+6cd\\
&-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-3abce\\
&-4bc^2e+9acde-3bcde+3c^2de-9abe^2-12bce^2+9cde^2\\
&+9bcdf-9bdef).
\end{aligned}
$$

With the condition that $d(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-3abce-4bc^2e+9acde-3bcde+3c^2de-9abe^2-12bce^2+9cde^2+9bcdf-9bdef)\neq 0$ and (56) hold, we have $J_{f_1(S^*)}\neq 0$ and the four zeros (57) of system (17) provide four periodic solutions of differential system (15). The set of conditions is not empty because the value $a=4, b=-16, c=-169/32, d=-4, f=-1, e=-1, h=1$ satisfy it.

If $a+d=0$ and $c-3d+3e=0$ we get $f_{12}=-3dh/4$=constant.

If $a+d=0$, $c-3d+3e\neq 0$ and $3a+c+3e=0$, $f_{14}=3(cd-ce-3e^2+3df)h/4(c-3d+3e)$=constant.

If $a+d \neq 0$ and $c-3d+3e=0$, solving $f_{12}=0$ we get $r=\sqrt{\dfrac{-2dh}{a+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ we obtain, if $c-3e+3f \neq 0$ $R=\sqrt{\dfrac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}$ and $\rho=\sqrt{\dfrac{2h(ac+3af-3ad-cd)}{(a+d)(c-3e+3f)}}$.

Conceding that

$$
\begin{array}{l}
a+d \neq 0, \quad c-3d+3e=0, \quad c-3e+3f \neq 0, \\
dh(a+d)<0, \quad h(3ad+cd-3ae)(a+d) \\
(c-3e+3f)>0 \quad \text{and} \quad h(ac+3af-3ad-cd) \\
(a+d)(c-3e+3f)>0.
\end{array}
\tag{58}
$$

System (17) for $l=1$ and $l=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(ac+3af-3ad-cd)}{(a+d)(c-3e+3f)}}$ given by

$$
\begin{array}{l}
\left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \dfrac{\pi}{2}\right), \\
\left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \dfrac{\pi}{2}\right), \\
\left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \dfrac{3\pi}{2}\right), \\
\left(\sqrt{\dfrac{-2dh}{a+d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(3ad+cd-3ae)}{(a+d)(c-3e+3f)}}, \dfrac{3\pi}{2}\right).
\end{array}
\tag{59}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \; & \frac{dh^4}{16(a+d)^4(c-3e+3f)^3}(3ad+cd \\
& -3ae)(-ac+3ad+cd-3af)(-b^2+2bc-c^2+9ad+6cd \\
& -18ae-6be-6ce-9e^2+9af+6bf+9df)(bc^2d+abce \\
& -3abde+3acde-4bcde+c^2de-3ace^2+3bcdf+3abef).
\end{aligned}
$$

With the condition that $(-b^2+2bc-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(bc^2d+abce-3abde+3acde-4bcde+c^2de-3ace^2+3bcdf+3abef) \neq 0$ and (58) we have $J_{f_1(S^*)} \neq 0$ and the four zeros (59) of system (17) provide four periodic solutions of differential system (15). The set of conditions is not empty because the value $a=3, b=-68, c=-3, d=-2, f=-6, e=-1, h=1$ satisfy it.

If $a+d \neq 0$, $c-3d+3e=0$ and $c-3e+3f=0$ we have $f_{14}=(cd+3ad-3ae)h/(4(a+d))=$ constant.

If $a+d \neq 0$ and $c-3d+3e \neq 0$, $r=\sqrt{\dfrac{3dR^2-6dh-cR^2-3eR^2}{3(a+d)}}$. Substituting r in f_{14} and solving $f_{14}=0$ if

$\Sigma_3 = c^2-9ad-6cd+18ae+6ce+9e^2-9af-9df \neq 0$

we obtain $R=\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}}$, $r=\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}}$ and $\rho=\sqrt{\dfrac{2h(c^2+3ce+9ae-9af)}{\Sigma_3}}$.

In case that

$$
\begin{array}{l}
\Sigma_3=c^2-9ad-6cd+18ae+6ce+9e^2-9af-9df\neq 0,\\
a+d\neq 0,\quad c-3d+3e\neq 0,\quad h(3ae-3ad-cd)\Sigma_3>0,\\
h(cd-ce-3e^2+3df)\Sigma_3>0\quad\text{and}\\
h(c^2+3ce+9ae-9af)\Sigma_3>0,
\end{array}
\tag{60}
$$

hold, system (17) for $l=1$ and $l=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(c^2+3ce+9ae-9af)}{\Sigma_3}}$ given by

$$
\begin{array}{l}
\left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{3\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{3\pi}{2}\right),\\
\left(\sqrt{\dfrac{6h(cd-ce-3e^2+3df)}{\Sigma_3}},\dfrac{3\pi}{2},\sqrt{\dfrac{6h(3ae-3ad-cd)}{\Sigma_3}},\dfrac{3\pi}{2}\right).
\end{array}
\tag{61}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)}= &\ \frac{9h^4}{16\Sigma_3^4}(3ad+cd-3ae)(c^2+9ae+3ce-9af)(cd-ce\\
&-3e^2+3df)(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce\\
&-9e^2+9af+6bf+9df)(3bc^2d-4bc^2e+9acde+3c^2de\\
&-9abe^2-9ace^2-12bce^2+9bcdf+9abef).
\end{aligned}
$$

Whenever $(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(3bc^2d-4bc^2e+9acde+3c^2de-9abe^2-9ace^2-12bce^2+9bcdf+9abef)\neq 0$ and (60) hold, then $J_{f_1(S^*)}\neq 0$ and the four zeros of system (17) provide four periodic solutions (61) of differential system (15). The set of conditions is not empty, the value $a=3, b=0, c=-4, d=-2, f=-2, e=-1, h=1$ satisfy it.

If $a+d\neq 0$, $c-3d+3e\neq 0$ and $\Sigma_3=0$ we have $f_{14}=h(cd+3ad-3ae)/(4(a+d))=$ constant.

Subcase 3.2: $\sin 2\alpha=0$.

Subcase 3.2.1: Assume that either $\alpha=0$ or $\alpha=\pi$. So if $f_{11}=-\dfrac{1}{8}crR^2\sin 2\beta$. If $f_{11}=0$ then consequently one of the following four subcases holds $c=0$, $r=0$ (studied in case 1), $R=0$, $\beta=p\pi/2$ with $p\in\mathbb{Z}$.

Subcase 3.2.1.1: $c=0$. No information as in subcase 1.1. So *in what follows in subcase* 3.2 *we assume that* $c\neq 0$.

Subcase 3.2.1.2: $R = 0$. Then

$$\begin{aligned} f_{12} = & -\frac{3}{8}\left[(a+2b+d)r^2 + 2(b+d)h\right], \\ f_{14} = & -\frac{1}{8}\left[3ar^2 + cr^2(2+\cos 2\beta) + (2h+r^2)(3b+2e+e\cos 2\beta)\right]. \end{aligned}$$

If $a+2b+d \neq 0$, then $f_{12} = 0 \Rightarrow r = \sqrt{\dfrac{-2h(b+d)}{a+2b+d}}$ and $\rho = \sqrt{\dfrac{2h(a+b)}{a+2b+d}}$. Substituting r in f_{14} and solving $f_{14} = 0$ we get $\beta = \pm\frac{1}{2}\arccos\Delta_5$, if $bc+cd-ae-be \neq 0$ where $\Delta_5 = (3b^2 - 2bc - 3ad - 2cd + 2ae + 2be)/(bc + cd - ae - be)$.

In the case that

$$\begin{array}{c} a+2b+d \neq 0, \quad bc+cd-ae-be \neq 0, \\ h(b+d)(a+2b+d) < 0, \quad h(a+b)(a+2b+d) > 0, \\ \text{and} \quad |\Delta_5| < 1, \end{array} \tag{62}$$

system (17) has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2h(a+b)}{a+2b+d}}$ given by

$$\begin{array}{c} \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}, 0, 0, \pm\dfrac{1}{2}\arccos\Delta_5\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}, \pi, 0, \pm\dfrac{1}{2}\arccos\Delta_5\right), \end{array} \tag{63}$$

which reduce to two zeros if $a+2b+d \neq 0$, $bc+cd-ae-be \neq 0$, $h(b+d)(a+2b+d) < 0$, $h(a+b)(a+2b+d) > 0$ and $|\Delta_5| = 1$.

The Jacobian is

$$\begin{aligned} J_{f_1(s*)} = & -\frac{9bh^4}{32(a+2b+d)^3}(a+b)(b+d)(3b^2 - bc - 3ad - cd + ae \\ & +be)(b^2 - bc - ad - cd + ae + be). \end{aligned}$$

Supposing that $b(3b^2 - bc - 3ad - cd + ae + be)(b^2 - bc - ad - cd + ae + be) \neq 0$ and (62) hold. This assumption is not empty because it is satisfied for the value $a = 2, b = -1, c = -2, d = -1, e = 0, h = -1$. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (63) of system (17) provide two periodic solutions of differential system (15) because when $R = 0$ the two solutions of β provide the same initial conditions.

If $a+2b+d = 0$, $f_{12} = -\dfrac{3h}{4}(b+d)$=constant.

If $a+2b+d \neq 0$ and $bc+cd-ae-be = 0$, $f_{14} = -h(3b^2 - 2bc - 3ad - 2cd + 2ae + 2be)/(4(a+2b+d))$ =constant.

Subcase 3.2.1.3: $\beta = \dfrac{p\pi}{2}$ with $p \in \mathbb{Z}$. Due to the periodicity of the sinus we study the subcases $p = 0$ and $p = 2$, and the subcases $p = 1$ and $p = 3$ together.

Subcase 3.2.1.3.1: Assume that either $p = 0$ or $p = 2$, i.e. $\beta = 0$ or $\beta = \pi$.

$$\begin{aligned} f_{12} = & -\frac{1}{8}\left[6(b+d)h + 3(a+2b+d)r^2 - 3(b-c+d-e)R^2\right], \\ f_{14} = & -\frac{1}{8}\left[6(b+e)h + 3(a+b+c+e)r^2 - 3(b-c+e-f)R^2\right]. \end{aligned}$$

If $b-c+d-e=0$ and $a+2b+d\neq 0$, solving $f_{12}=0$, we obtain $r=\sqrt{-\dfrac{2h(b+d)}{a+2b+d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R if $b-c+e-f\neq 0$ we get $R=\sqrt{\dfrac{2hN_1}{\delta_1}}$ and $\rho=\sqrt{\dfrac{2hN_2}{\delta_1}}$ where $N_1=b^2-cd+be-bc+ae-ad$, $N_2=cd+ab-ac+ad-af-bf$ and $\delta_1=(a+2b+d)(b-c+e-f)$.

Assuming that

$$\begin{array}{l} b-c+d-e=0, \quad a+2b+d\neq 0, \quad b-c+e-f\neq 0, \\ h(b+d)(a+2b+d)<0, \quad hN_1\delta_1>0 \quad \text{and} \quad hN_2\delta_1>0. \end{array} \tag{64}$$

System (17) for $p=0$ and $p=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2hN_2}{\delta_1}}$ given by

$$\begin{array}{l} \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},0,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},0\right), \\ \left(\sqrt{-\dfrac{2h(b+d)}{a+2b+d}},\pi,\sqrt{\dfrac{2hN_1}{\delta_1}},\pi\right). \end{array} \tag{65}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(s*)} = \; & -\frac{9h^4(b+d)}{16(a+2b+d)^4(b-c+e-f)^3}N_1N_2(b^2-2bc+c^2-ad \\ & -2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c \\ & -b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de \\ & +ace^2+bce^2+b^2cf+bcdf-abef-b^2ef). \end{aligned}$$

With the condition $(b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df)(b^2c^2-b^3c-b^2cd+bc^2d+ab^2e-abce-bc^2e+abde-acde-c^2de+ace^2+bce^2+b^2cf+bcdf-abef-b^2ef)\neq 0$ and (64), we have $J_{f_1(S^*)}\neq 0$. The set of conditions is not empty because it is satisfied for the value $a=1,b=2,c=-1,d=-4,e=-1,f=-3,h=1$. Therefore the four zeros (65) of system (17) provide four periodic solutions of differential system (15).

If $b-c+d-e=0$ and $a+2b+d=0$, $f_{12}=-\dfrac{3h}{4}(b+d)=$ constant.

If $b-c+d-e=0$, $a+2b+d\neq 0$ and $b-c+e-f=0$ we have $f_{14}=3h(-b^2+bc+ad+cd-ae-be)/(4(a+2b+d))=$ constant.

If $b-c+d-e\neq 0$ and $a+2b+d=0$, solving $f_{12}=0$, we obtain $R=\sqrt{\dfrac{2h(b+d)}{b-c+d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r if $a+b+c+e\neq 0$ we get $r=\sqrt{\dfrac{2hN_3}{\delta_2}}$ and $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ where $N_3=e^2-cd+ce+be-bf-df$, $N_4=-ac-ae-cb-c^2-cd-ce-bf-df$ and $\delta_2=(b-c+d-e)(a+b+c+e)$.

Assuming that

$$(66)\qquad \begin{array}{l} b-c+d-e\neq 0,\quad a+2b+d=0,\quad a+b+c+e\neq 0,\\ h(b+d)(b-c+d-e)>0,\quad hN_3\delta_2>0\quad \text{and}\quad hN_4\delta_2>0. \end{array}$$

System (17) for $p=0$ and $p=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2hN_4}{\delta_2}}$ given by

$$(67)\qquad \begin{array}{c} \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},0\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},0\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},0,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},\pi\right),\\ \left(\sqrt{\dfrac{2hN_3}{\delta_2}},\pi,\sqrt{\dfrac{2h(b+d)}{b-c+d-e}},\pi\right). \end{array}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(s*)} = & -\frac{9h^4(b+d)}{16(b-c+d-e)^4(a+b+c+e)^3}N_3N_4(2bc-b^2-c^2\\ & +ad+2cd-2ae-2be-2ce-e^2+af+2bf+df)(b^2ce\\ & -bc^2d+bc^2e+acde+c^2de-abe^2+bce^2+cde^2-b^2cf\\ & -bcdf-b^2ef-bdef). \end{aligned}$$

With the condition $(2bc-b^2-c^2+ad+2cd-2ae-2be-2ce-e^2+af+2bf+df)(b^2ce-bc^2d+bc^2e+acde+c^2de-abe^2+bce^2+cde^2-b^2cf-bcdf-b^2ef-bdef)\neq 0$ and (66) we have $J_{f_1(S^*)}\neq 0$. The set of conditions is not empty because it is satisfied for the value $a=7, b=-2, c=-1, d=-3, e=-1, f=-2, h=1$. Therefore the four zeros (67) of system (17) provide four periodic solutions of differential system (15).

If $b-c+d-e\neq 0$, $a+2b+d=0$ and $a+b+c+e=0$ we have $f_{14}=-3h(bf+cd+df-be-ce-e^2)/(4(b-c+d-e))=$ constant.

If $b-c+d-e\neq 0$ and $a+2b+d\neq 0$, solving $f_{12}=0$ we get $R=\sqrt{\dfrac{2h(b+d)+(a+2b+d)r^2}{(b-c+d-e)}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r if

$$\omega=b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-af-2bf-df\neq 0,$$

we get $r=\sqrt{\dfrac{2hN_5}{\omega}}$, $R=\sqrt{\dfrac{2hN_6}{\omega}}$ and $\rho=\sqrt{\dfrac{2hN_7}{\omega}}$, where $N_5=cd-be-ce-e^2+bf+df$, $N_6=b^2-ad-cd+ae-bc+be$, $N_7=c^2+ce+ae-af-bc-bf$.

In the case that

$$(68)\qquad \begin{array}{l} \omega=b^2-2bc+c^2-ad-2cd+2ae+2be+2ce+e^2-\\ \qquad af-2bf-df\neq 0,\\ b-c+d-e\neq 0,\quad a+2b+d\neq 0,\quad hN_5\omega>0,\\ hN_6\omega>0\quad \text{and}\quad hN_7\omega>0, \end{array}$$

hold, then system (17) for $p = 0$ and $p = 2$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2hN_7}{\omega}}$ given by

$$
\begin{gathered}
\left(\sqrt{\frac{2hN_5}{\omega}}, 0, \sqrt{\frac{2hN_6}{\omega}}, 0\right),\\
\left(\sqrt{\frac{2hN_5}{\omega}}, 0, \sqrt{\frac{2hN_6}{\omega}}, \pi\right),\\
\left(\sqrt{\frac{2hN_5}{\omega}}, \pi, \sqrt{\frac{2hN_6}{\omega}}, 0\right),\\
\left(\sqrt{\frac{2hN_5}{\omega}}, \pi, \sqrt{\frac{2hN_6}{\omega}}, \pi\right).
\end{gathered} \tag{69}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \ & \frac{9h^4}{16\omega^3} N_5 N_6 N_7 (bc^2 d - b^2 ce - bc^2 e - acde - c^2 de + abe^2 \\
& + ace^2 + bce^2 + b^2 cf + bcdf - abef - b^2 ef).
\end{aligned}
$$

Conceding that $(bc^2d - b^2ce - bc^2e - acde - c^2de + abe^2 + ace^2 + bce^2 + b^2cf + bcdf - abef - b^2ef) \neq 0$ and (68) hold. This assumption is not empty because the value $a = 4, b = -3, c = -2, d = -2, f = -5, e = -1, h = -1$ satisfy it. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (69) of system (17) provide four periodic solutions of differential system (15).

If $b - c + d - e \neq 0$, $a + 2b + d \neq 0$ and $\omega = 0$ we have $f_{14} = -3h(cd - be - ce - e^2 + bf + df)/(4(b - c + d - e))$=constant.

Subcase 3.2.1.3.2: Assume that either $p = 1$ or $p = 3$, i.e. $\beta = \dfrac{\pi}{2}$ or $\beta = \dfrac{3\pi}{2}$.

$$
\begin{aligned}
f_{12} = \ & -\frac{1}{8}\big[6(b + d)h + 3(a + 2b + d)r^2 - (3b - c + 3d - e)R^2\big],\\
f_{14} = \ & -\frac{1}{8}\big[2h(3b + e) + (3a + 3b + c + e)r^2 - (3b - c + e - 3f)R^2\big].
\end{aligned}
$$

If $3b - c + 3d - e = 0$ and $a + 2b + d \neq 0$, solving $f_{12} = 0$ we get $r = \sqrt{-2h(b + d)/(a + 2b + d)}$. Substituting r in f_{14} and solving $f_{14} = 0$ if $3b - c + e - 3f \neq 0$ we obtain $R = \sqrt{\dfrac{2hN_8}{\delta_3}}$ where $\delta_3 = (a + 2b + d)(3b - c + e - 3f)$, $N_8 = 3b^2 - cd - bc + be - 3ad + ae$ and $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ where $N_9 = (3ab - ac + 3ad + cd - 3af - 3bf)$.

If we have

$$
\begin{gathered}
3b - c + 3d - e = 0, \quad a + 2b + d \neq 0, \quad 3b - c + e - 3f \neq 0,\\
h(b + d)(a + 2b + d) < 0, \quad h\delta_3 N_8 > 0 \quad \text{and} \quad h\delta_3 N_9 > 0,
\end{gathered} \tag{70}
$$

then system (17) for $p = 1$ and $p = 3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2hN_9}{\delta_3}}$ given by

$$\begin{aligned}
&\left(\sqrt{\frac{-2h(b+d)}{(a+2b+d)}}, 0, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{-2h(b+d)}{(a+2b+d)}}, 0, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{-2h(b+d)}{(a+2b+d)}}, \pi, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{-2h(b+d)}{(a+2b+d)}}, \pi, \sqrt{\frac{2hN_8}{\delta_3}}, \frac{3\pi}{2}\right).
\end{aligned} \tag{71}$$

The Jacobian is

$$\begin{aligned}
J_{f_1(s*)} = \; & -\frac{h^4(b+d)N_8N_9}{16(a+2b+d)^4(3b-c+e-3f)^3}(9b^2-6bc+c^2-9ad\\
& -6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(b^2c^2\\
& -3b^3c-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde\\
& +3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef\\
& -3b^2ef).
\end{aligned}$$

Supposing that $(9b^2-6bc+c^2-9ad-6cd+6ae+6be+2ce+e^2-9af-18bf-9df)(-3b^3c+b^2c^2-3b^2cd+bc^2d+3ab^2e-abce-4b^2ce+bc^2e+3abde+3acde+c^2de-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef) \neq 0$ and (70) hold. This assumption is not empty because it is satisfied for the value $a = 0, b = 13/3, c = -4, d = -6, f = -2, e = -1, h = 1$. As a result of that, we have $J_{f_1(S^*)} \neq 0$ and the four zeros (71) of system (17) provide four periodic solutions of differential system (15).

If $3b - c + 3d - e = 0$ and $a + 2b + d = 0$, $f_{12} = -\dfrac{3h}{4}(b + d)$=constant.

If $3b - c + 3d - e = 0$, $a + 2b + d \neq 0$ and $3b - c + e - 3f = 0$ we have $f_{14} = (cd - 3b^2 + bc - be + 3ad - ae)h/(4(a + 2b + d))$=constant.

If $3b - c + 3d - e \neq 0$ and $a + 2b + d = 0$, solving $f_{12} = 0$ we get $R = \sqrt{\dfrac{6h(b+d)}{3b-c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ if $3a+3b+c+e \neq 0$ we obtain $r = \sqrt{\dfrac{2hN_{10}}{\delta_4}}$ where $\delta_4 = (3a + 3b + c + e)(3b - c + 3d - e)$, $N_{10} = 3be + ce + e^2 - 3cd - 9bf - 9df$ and $\rho = \sqrt{\dfrac{2hN_{11}}{\delta_4}}$ where $N_{11} = (3ac + 3ae + 3bc + c^2 + 3cd + ce + 9bf + 9df)$.

If we have

$$\begin{aligned}
&3b-c+3d-e \neq 0, \quad a+2b+d = 0,\\
&3a+3b+c+e \neq 0, h(b+d)(3b-c+3d-e) > 0,\\
&h\delta_4 N_{10} > 0 \quad \text{and} \quad h\delta_4 N_{11} > 0,
\end{aligned} \tag{72}$$

then system (17) for $p = 1$ and $p = 3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\frac{2hN_{11}}{\delta_4}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, 0, \sqrt{\frac{6h(b+d)}{3b-c+3d-e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\frac{6h(b+d)}{3b-c+3d-e}}, \frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, 0, \sqrt{\frac{6h(b+d)}{3b-c+3d-e}}, \frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{10}}{\delta_4}}, \pi, \sqrt{\frac{6h(b+d)}{3b-c+3d-e}}, \frac{3\pi}{2}\right).
\end{aligned} \tag{73}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \quad & \frac{9h^4(b+d)N_{10}N_{11}}{16(3b-c+3d-e)^4(3a+3b+c+e)^3}(6bc-9b^2-c^2\\
& +9ad+6cd-6ae-6be-2ce-e^2+9af+18bf+9df)\\
& (bc^2d+4abce+3b^2ce+bc^2e+3acde+4bcde+c^2de+abe^2\\
& +bce^2+cde^2+3b^2cf+3bcdf+3b^2ef+3bdef).
\end{aligned}
$$

Supposing that $(6bc-9b^2-c^2+9ad+6cd-6ae-6be-2ce-e^2+9af+18bf+9df)(bc^2d+4abce+3b^2ce+bc^2e+3acde+4bcde+c^2de+abe^2+bce^2+cde^2+3b^2cf+3bcdf+3b^2ef+3bdef) \neq 0$ and (72) hold. This assumption is not empty because it is satisfied for the value $a = 4, b = -2, c = -1, d = 0, f = -1/2, e = -1, h = 1$. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (73) of system (17) provide four periodic solutions of differential system (15).

If $3b - c + 3d - e \neq 0$, $a + 2b + d = 0$ and $3a + 3b + c + e = 0$ we have $f_{14} = (-3cd + 3be + ce + e^2 - 9bf - 9df)h/4(3b - c + 3d - e)$=constant.

If $3b - c + 3d - e \neq 0$ and $a + 2b + d \neq 0$ we have $R = \sqrt{\frac{6h(b+d)+3(a+2b+d)r^2}{3b-c+3d-e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ if

$\omega_1 = 9b^2 - 6bc + c^2 - 9ad - 6cd + 6ae + 6be + 2ce + e^2 - 9af - 18bf - 9df \neq 0$ so $r = \sqrt{\frac{2hN_{12}}{\omega_1}}$, $R = \sqrt{\frac{6hN_{13}}{\omega_1}}$ and $\rho = \sqrt{\frac{2hN_{14}}{\omega_1}}$ where $N_{12} = 3cd - 3be - ce - e^2 + 9bf + 9df$, $N_{13} = 3b^2 - 3ad - cd + ae - bc + be$, $N_{14} = c^2 - 3bc + 3ae + ce - 9af - 9bf$.

Supposing that

$$
\begin{aligned}
&\omega_1 = 9b^2 - 6bc + c^2 - 9ad - 6cd + 6ae + 6be + 2ce + e^2 - 9af\\
&-18bf - 9df \neq 0, \quad 3b - c + 3d - e \neq 0, \quad a + 2b + d \neq 0,\\
&hN_{12}\omega_1 > 0, \quad hN_{13}\omega_1 > 0 \quad \text{and} \quad hN_{14}\omega_1 > 0.
\end{aligned} \tag{74}
$$

System (17) for $p=1$ and $p=3$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2hN_{14}}{\omega_1}}$ given by

$$
\begin{aligned}
&\left(\sqrt{\frac{2hN_{12}}{\omega_1}},0,\sqrt{\frac{6hN_{13}}{\omega_1}},\frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{12}}{\omega_1}},0,\sqrt{\frac{6hN_{13}}{\omega_1}},\frac{3\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{12}}{\omega_1}},\pi,\sqrt{\frac{6hN_{13}}{\omega_1}},\frac{\pi}{2}\right),\\
&\left(\sqrt{\frac{2hN_{12}}{\omega_1}},\pi,\sqrt{\frac{6hN_{13}}{\omega_1}},\frac{3\pi}{2}\right).
\end{aligned}
\tag{75}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \;& -\frac{9h^4}{16\omega_1^3}N_{12}N_{13}N_{14}(bc^2d-5b^2ce+bc^2e+3acde+c^2de+abe^2\\
& -ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef).
\end{aligned}
$$

Assuming that $(bc^2d-5b^2ce+bc^2e+3acde+c^2de+abe^2-ace^2-bce^2+3b^2cf+3bcdf-3abef-3b^2ef)\neq 0$ and (74) hold. This set of conditions is not empty because the value $a=7,b=-6,c=-24,d=-45,f=17,e=-1,h=-1$ satisfy it. Then $J_{f_1(S^*)}\neq 0$ and the four zeros (75) of system (17) provide four periodic solutions of differential system (15).

If $3b-c+3d-e\neq 0$, $a+2b+d\neq 0$ and $\omega_1=0$ then $f_{14}=-hN_{12}/(4(3b-c+3d-e))=$ constant.

Subcase 3.2.2: Assuming either $\alpha=\dfrac{\pi}{2}$ or $\alpha=\dfrac{3\pi}{2}$. So $f_{11}=-\dfrac{1}{8}crR^2\sin 2\beta$. If $f_{11}=0$ then consequently one of the following four subcases holds $c=0$, $r=0$ (studied in case 1), $R=0$, $\beta=q\pi/2$ with $q\in\mathbb{Z}$.

Subcase 3.2.2.1: $c=0$. No information about the periodic orbits as in subcase 1.1. So *in what follows in subcase* 3.2.2 *we assume that* $c\neq 0$.

Subcase 3.2.2.2: $R=0$. Then

$$
\begin{aligned}
f_{12} = \;& -\frac{1}{8}\left[(3a+2b+3d)r^2+2(b+3d)h\right],\\
f_{14} = \;& -\frac{1}{8}\big[(3a+b+2c+2e+(c-e)\cos 2\beta)r^2+2h(b+2e-\\
& e\cos 2\beta)\big].
\end{aligned}
$$

If $3a+2b+3d\neq 0$, solving $f_{12}=0$ we get $r=\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}$ and $\rho=\sqrt{\dfrac{2h(3a+b)}{3a+2b+3d}}$. Substituting r in f_{14} and solving $f_{14}=0$ we obtain, if $3cd+3ae+bc+be\neq 0$, $\beta=\pm\dfrac{1}{2}\arccos\Delta_6$, where $\Delta_6=\dfrac{b^2-2bc-9ad-6cd+6ae+2be}{3cd+3ae+bc+be}$.

Whenever

$$(76)\qquad \begin{array}{l} 3a+2b+3d\neq 0, \quad 3cd+3ae+bc+be\neq 0,\\ h(b+3d)(3a+2b+3d)<0, \quad h(3a+b)(3a+2b+3d)>0\\ \quad \text{and} \quad |\Delta_6|<1, \end{array}$$

system (17) has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{2h(3a+b)}{3a+2b+3d}}$ given by

$$(77)\qquad \begin{array}{l} \left(\sqrt{\dfrac{-2h(b+3d)}{3a+2b+3d}},\dfrac{\pi}{2},0,\pm\dfrac{1}{2}\arccos\Delta_6\right),\\ \left(\sqrt{\dfrac{-2h(b+3d)}{3a+2b+3d}},\dfrac{3\pi}{2},0,\pm\dfrac{1}{2}\arccos\Delta_6\right), \end{array}$$

which reduce to two zeros if $3a+2b+3d\neq 0$, $3cd+3ae+bc+be\neq 0$, $h(b+3d)(3a+2b+3d)<0$, $h(3a+b)(3a+2b+3d)>0$ and $|\Delta_6|=1$.

The Jacobian is

$$\begin{aligned} J_{f_1(s*)} = \;& \frac{bh^4}{32(3a+2b+3d)^3}(3a+b)(b+3d)(b^2-3bc-9ad-9cd\\ &+3ae+be)(b^2-bc-9ad-3cd+9ae+3be). \end{aligned}$$

Assuming that $b(b^2-3bc-9ad-9cd+3ae+be)(b^2-bc-9ad-3cd+9ae+3be)\neq 0$ and (76) hold. This supposition is not empty because the value $a=1/2, b=-1, c=-3/8, d=0, e=0, h=-1$. Therefore $J_{f_1(S^*)}\neq 0$ and the four zeros (77) of system (17) provide only two periodic solutions of differential system (15) because when $R=0$ the two solutions of β provide the same initial conditions in (10).

If $3a+2b+3d=0$, $f_{12}=-\dfrac{h}{4}(b+3d)$=constant.

If $3a+2b+3d\neq 0$ and $3cd+3ae+bc+be=0$, we have $f_{14}=-\dfrac{h(b^2-2bc-9ad-6cd+6ae+2be)}{4(3a+2b+3d)}$= constant.

Subcase 3.2.2.3: $\beta=\dfrac{q\pi}{2}$ with $q\in\mathbb{Z}$. Then due to the periodicity of the sinus we study the subcases $q=0$ and $q=2$, and the subcases $p=1$ and $p=3$ together.

Subcase 3.2.2.3.1: Assume that either $q=0$ or $q=2$, i.e. either $\beta=0$ or $\beta=\pi$.

$$\begin{aligned} f_{12} = &\; -\frac{1}{8}\left[2(b+3d)h+(3a+2b+3d)r^2-(b-3c+3d-e)R^2\right],\\ f_{14} = &\; -\frac{1}{8}\left[2(b+e)h+(3a+b+3c+e)r^2-(b-3c+e-3f)R^2\right]. \end{aligned}$$

If $b-3c+3d-e=0$ and $3a+2b+3d\neq 0$, solving $f_{12}=0$, we obtain $r=\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}$. Substituting r in f_{14} and solving $f_{14}=0$ with respect to R if $b-3c+e-3f\neq 0$ we get $R=\sqrt{\dfrac{2h(b^2-9ad-9cd+3ae-3bc+be)}{(3a+2b+3d)(b-3c+e-3f)}}$ and

$$\rho=\sqrt{\frac{6h(3cd-bf+ab-3ac+3ad-3af)}{(3a+2b+3d)(b-3c+e-3f)}}.$$

Conceding that

$$
\begin{aligned}
& b-3c+3d-e=0, \quad 3a+2b+3d\neq 0, \quad b-3c+e-3f\neq 0\\
& h(b^2-9ad-9cd+3ae-3bc+be)(3a+2b+3d)\\
& (b-3c+e-3f)>0, \quad h(b+3d)(3a+2b+3d)<0 \quad \text{and}\\
& h(3cd-bf+ab-3ac+3ad-3af)(3a+2b+3d)(b-3c+e-3f)>0.
\end{aligned} \tag{78}
$$

System (17) for $q=0$ and $q=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{6h(3cd-bf+ab-3ac+3ad-3af)}{(3a+2b+3d)(b-3c+e-3f)}}$ given by

$$
\begin{aligned}
&\left(\sqrt{-\frac{2h(b+3d)}{3a+2b+3d}},\frac{\pi}{2},\sqrt{\frac{2h(b^2-9ad-9cd+3ae-3bc+be)}{(3a+2b+3d)(b-3c+e-3f)}},0\right),\\
&\left(\sqrt{-\frac{2h(b+3d)}{3a+2b+3d}},\frac{\pi}{2},\sqrt{\frac{2h(b^2-9ad-9cd+3ae-3bc+be)}{(3a+2b+3d)(b-3c+e-3f)}},\pi\right),\\
&\left(\sqrt{-\frac{2h(b+3d)}{3a+2b+3d}},\frac{3\pi}{2},\sqrt{\frac{2h(b^2-9ad-9cd+3ae-3bc+be)}{(3a+2b+3d)(b-3c+e-3f)}},0\right),\\
&\left(\sqrt{-\frac{2h(b+3d)}{3a+2b+3d}},\frac{3\pi}{2},\sqrt{\frac{2h(b^2-9ad-9cd+3ae-3bc+be)}{(3a+2b+3d)(b-3c+e-3f)}},\pi\right).
\end{aligned} \tag{79}
$$

Its Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \; & -\frac{3h^4(b+3d)(b^2-3bc-9ad-9cd+3ae+be)}{16(3a+2b+3d)^4(b-3c+e-3f)^3}(ab-3ac\\
& +3ad+3cd-3af-bf)(b^2-6bc+9c^2-9ad-18cd\\
& +6ae+2be+6ce+e^2-9af-6bf-9df)(b^3c-3b^2c^2\\
& +3b^2cd-9bc^2d+3ab^2e-9abce+3bc^2e+9abde+9acde\\
& +9c^2de+12bcde-3ace^2-bce^2-3b^2cf-9bcdf-9abef\\
& -3b^2ef).
\end{aligned}
$$

In the case that $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(b^3c-3b^2c^2+3b^2cd-9bc^2d+3ab^2e-9abce+3bc^2e+9abde+9acde+9c^2de+12bcde-3ace^2-bce^2-3b^2cf-9bcdf-9abef-3b^2ef)\neq 0$ and (78) hold, then $J_{f_1(S^*)}\neq 0$ and the four zeros (79) of system (17) provide four periodic solutions of differential system (15). The set of conditions in not empty because for the value $a=1, b=5, c=-2, d=-4, f=-1, e=-1, h=1$ it is satisfied.

If $b-3c+3d-e=0$ and $3a+2b+3d=0$, $f_{12}=-\dfrac{1}{4}(b+3d)h$=constant.

If $b-3c+3d-e=0$, $3a+2b+3d\neq 0$ and $b-3c+e-3f=0$, we have $f_{14}=(-b^2+3bc+9ad+9cd-3ae-be)/(4(3a+2b+3d))$

If $b-3c+3d-e\neq 0$ and $3a+2b+3d=0$, solving $f_{12}=0$, we obtain $R=\sqrt{\dfrac{2h(b+3d)}{b-3c+3d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r if $3a+b+3c+e\neq 0$ we get $r=\sqrt{\dfrac{2hN_{15}}{\delta_5}}$ and $\rho=\sqrt{\dfrac{-6hN_{16}}{\delta_5}}$ where $N_{15}=e^2-9cd+be+3ce-3bf-9df$, $N_{16}=3ac+ae+bc+3c^2+3cd+ce+bf+3df$ and $\delta_5=(b-3c+3d-e)(3a+b+3c+e)$.

Supposing that

$$\begin{aligned} b-3c+3d-e\neq 0, \quad 3a+2b+3d=0,\\ 3a+b+3c+e\neq 0, \quad h(b+3d)(b-3c+3d-e)>0,\\ h\delta_5 N_{15}>0 \quad \text{and} \quad h\delta_5 N_{16}<0. \end{aligned} \tag{80}$$

System (17) for $q=0$ and $q=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{-6hN_{16}}{\delta_5}}$ given by

$$\begin{aligned} &\left(\sqrt{\frac{2hN_{15}}{\delta_5}},\frac{\pi}{2},\sqrt{\frac{2h(b+3d)}{b-3c+3d-e}},0\right),\\ &\left(\sqrt{\frac{2hN_{15}}{\delta_5}},\frac{\pi}{2},\sqrt{\frac{2h(b+3d)}{b-3c+3d-e}},\pi\right),\\ &\left(\sqrt{\frac{2hN_{15}}{\delta_5}},\frac{3\pi}{2},\sqrt{\frac{2h(b+3d)}{b-3c+3d-e}},0\right),\\ &\left(\sqrt{\frac{2hN_{15}}{\delta_5}},\frac{3\pi}{2},\sqrt{\frac{2h(b+3d)}{b-3c+3d-e}},\pi\right). \end{aligned} \tag{81}$$

Its Jacobian is

$$\begin{aligned} J_{f_1(s*)} = \;& \frac{3h^4(b+3d)N_{15}N_{16}}{16(b-3c+3d-e)^4(3a+b+3c+e)^3)}(b^2-6bc+9c^2\\ & -9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)\\ & (12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+9c^2de\\ & +3abe^2+5bce^2+3cde^2-3b^2cf-9bcdf+3b^2ef+9bdef). \end{aligned}$$

Supposing that $(b^2-6bc+9c^2-9ad-18cd+6ae+2be+6ce+e^2-9af-6bf-9df)(12abce-9bc^2d+5b^2ce+15bc^2e+9acde+12bcde+9c^2de+3abe^2+5bce^2+3cde^2-3b^2cf-9bcdf+3b^2ef+9bdef)\neq 0$ and (80) hold. This assumption is not empty because the value $a=11/3, b=-1, c=-2, d=-3, f=0, e=-1, h=1$ satisfy it. Therefore $J_{f_1(S^*)}\neq 0$ and the four zeros (81) of system (17) provide four periodic solutions of differential system (15).

If $b-3c+3d-e\neq 0$, $3a+2b+3d=0$ and $3a+b+3c+e=0$ then $f_{14}=h(eb+e^2-9cd+3ce-3fb-9fd)/(4(b-3c+3d-e))$=constant.

If $b-3c+3d-e\neq 0$ and $3a+2b+3d\neq 0$, solving $f_{12}=0$ we get $R=\sqrt{\dfrac{2h(b+3d)+(3a+2b+3d)r^2}{b-3c+3d-e}}$. Substituting R in f_{14} and solving $f_{14}=0$ with respect to r if $\omega_2=b^2+9c^2-9ad+6ae+e^2-18cd+6ce-6bc+2be-6bf-9af-9df\neq 0$ we get $r=\sqrt{\dfrac{2hN_{17}}{\omega_2}}$, $R=\sqrt{\dfrac{2hN_{18}}{\omega_2}}$ and $\rho=\sqrt{\dfrac{6hN_{19}}{\omega_2}}$ where $N_{16}=9cd-be-3ce-e^2+3bf+9df$, $N_{17}=b^2-9ad-9cd+3aeb-3bc+be$ and $N_{18}=3c^2+ce+ae-3af-bc-bf$. Conceding that

$$\begin{aligned} b-3c+3d-e\neq 0, \quad 3a+2b+3d\neq 0, \quad \omega_2\neq 0,\\ hN_{17}\omega_2>0, \quad hN_{18}\omega_2>0 \quad \text{and} \quad hN_{19}\omega_2>0. \end{aligned} \tag{82}$$

System (17) for $q=0$ and $q=2$ has four zeros $S^*=(r,\alpha,R,\beta)$ with $\rho=\sqrt{\dfrac{6hN_{19}}{\omega_2}}$ given by

$$
\begin{array}{c}
\left(\sqrt{\dfrac{2hN_{17}}{\omega_2}},\dfrac{\pi}{2},\sqrt{\dfrac{2hN_{18}}{\omega_2}},0\right),\\
\left(\sqrt{\dfrac{2hN_{17}}{\omega_2}},\dfrac{\pi}{2},\sqrt{\dfrac{2hN_{18}}{\omega_2}},\pi\right),\\
\left(\sqrt{\dfrac{2hN_{17}}{\omega_2}},\dfrac{3\pi}{2},\sqrt{\dfrac{2hN_{18}}{\omega_2}},0\right),\\
\left(\sqrt{\dfrac{2hN_{17}}{\omega_2}},\dfrac{3\pi}{2},\sqrt{\dfrac{2hN_{18}}{\omega_2}},\pi\right).
\end{array}
\tag{83}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)}= & -\frac{9h^4}{16\omega_2^3}N_{17}N_{18}N_{19}(5bc^2e-3bc^2d-b^2ce+3acde+3c^2de\\
& +abe^2-ace^2+bce^2-b^2cf-3bcdf-3abef-b^2ef).
\end{aligned}
$$

Whenever $(5bc^2e-3bc^2d-b^2ce+3acde+3c^2de+abe^2-ace^2+bce^2-b^2cf-3bcdf-3abef-b^2ef)\neq 0$ and (82) hold, $J_{f_1(S^*)}\neq 0$. Therefore the four zeros (83) of (17) provide four periodic solutions of (15). The set of conditions is not empty because the value $a=24801, b=-\dfrac{1256502183358496925170226771841 47}{73874071844355067904605238067 2}, c=-4232808, d=-23930, e=-72913, f=-92159, h=1$ satisfy it.

If $b-3c+3d-e\neq 0$, $3a+2b+3d\neq 0$ and $\omega_2=0$ we have $\omega_2=0$ we have $f_{14}=h(eb+e^2-9cd+3ce-3bf=$ constant.

Subcase 3.2.2.3.2: Assume that either $q=1$ or $q=3$, i.e. either $\beta=\dfrac{\pi}{2}$ or $\beta=\dfrac{3\pi}{2}$. Then

$$
\begin{aligned}
f_{12}= & -\frac{1}{8}\big[2h(b+3d)+(3a+2b+3d)r^2+(c-b-3d+3e)R^2\big],\\
f_{14}= & -\frac{1}{8}\big[2h(b+3e)+(3a+b+c+3e)r^2+(3f-b-3e+c)R^2\big].
\end{aligned}
$$

If $c-b-3d+3e=0$ and $3a+2b+3d\neq 0$, solving $f_{12}=0$, we obtain $r=\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}$. Substituting r in f_{14} and solving $f_{14}=0$, if $b-c+3e-3f\neq 0$, we obtain $R=\sqrt{\dfrac{2hN_{20}}{\delta_6}}$ and $\rho=\sqrt{\dfrac{6hN_{21}}{\delta_6}}$, where $N_{20}=b^2-bc-9ad-3cd+9ae+3be$, $N_{21}=cd+ab-ac+3ad-3af-bf$ and $\delta_6=(3a+2b+3d)(b-c+3e-3f)$.

Supposing that

$$
\begin{array}{l}
c-b-3d+3e=0, 3a+2b+3d\neq 0,\\
b-c+3e-3f\neq 0,\quad h(b+3d)(3a+2b+3d)<0,\\
h\delta_6N_{20}>0 \quad\text{and}\quad h\delta_6N_{21}>0.
\end{array}
\tag{84}
$$

System (17) for $q = 1$ and $q = 3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{6hN_{21}}{\delta_6}}$ given by

$$
(85)\qquad
\begin{array}{l}
\left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{20}}{\delta_6}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{20}}{\delta_6}}, \dfrac{3\pi}{2}\right),\\
\left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{20}}{\delta_6}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{-\dfrac{2h(b+3d)}{3a+2b+3d}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{20}}{\delta_6}}, \dfrac{3\pi}{2}\right).
\end{array}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \; & \frac{3h^4(b+3d)N_{20}N_{21}}{16(3a+2b+3d)^4(b-c+3e-3f)^3}(b^2-2bc+c^2-9ad\\
& -6cd+18ae+6be+6ce+9e^2-9af-6bf-9df)(b^3c\\
& -b^2c^2+3b^2cd-3bc^2d+3ab^2e-3abce+4b^2ce-bc^2e\\
& +9abde-9acde+12bcde-3c^2de+9ace^2+3bce^2-3b^2cf\\
& -9bcdf-9abef-3b^2ef).
\end{aligned}
$$

Assuming that $(b^2-2bc+c^2-9ad-6cd+18ae+6be+6ce+9e^2-9af-6bf-9df)(b^3c-b^2c^2+3b^2cd-3bc^2d+3ab^2e-3abce+4b^2ce-bc^2e+9abde-9acde+12bcde-3c^2de+9ace^2+3bce^2-3b^2cf-9bcdf-9abef-3b^2ef) \neq 0$ and (84) hold, then $J_{f_1(S^*)} \neq 0$ and the four zeros (85) of system (17) provide four periodic solutions of differential system (15). The set of conditions is not empty because the value $a = 2, b = 1, c = -2, d = -2, f = -4, e = -1, h = 1$.

If $3a+2b+3d = 0$ and $c-b-3d+3e = 0$ we get $f_{12} = -\dfrac{1}{4}(b+3d)h =$constant.

If $c-b-3d+3e = 0$, $b-c+3e-3f \neq 0$ and $b-c+3e-3f = 0$ we have $f_{14} = -N_{20}/(4(3a+2b+3d)) = constant$.

If $c-b-3d+3e \neq 0$ and $3a+2b+3d = 0$, we have $R = \sqrt{\dfrac{2h(b+3d)}{b-c+3d-3e}}$. Substituting R in f_{14} and solving $f_{14} = 0$ if $3a+b+c+3e \neq 0$, we get $r = \sqrt{\dfrac{6hN_{22}}{\delta_7}}$ and $\rho = \sqrt{-\dfrac{2hN_{23}}{\delta_7}}$, where $N_{22} = be-bf+ce-cd+3e^2-3df$, $N_{23} = 3ac+bc+c^2+3cd+9ae+3ce+3bf+9df$ and $\delta_7 = (b-c+3d-3e)(3a+b+c+3e)$.

Assuming that

$$
(86)\qquad
\begin{array}{l}
c-b-3d+3e \neq 0, \quad 3a+2b+3d = 0, \quad 3a+b+c+3e \neq 0\\
h(b+3d)(b-c+3d-3e) > 0, \quad h\delta_7 N_{22} > 0 \quad \text{and} \quad h\delta_7 N_{23} < 0.
\end{array}
$$

System (17) for $q = 1$ and $q = 3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{-\dfrac{2hN_{23}}{\delta_7}}$ given by

$$
\begin{array}{c}
\left(\sqrt{\dfrac{6hN_{22}}{\delta_7}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(b+3d)}{b-c+3d-3e}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{6hN_{22}}{\delta_7}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2h(b+3d)}{b-c+3d-3e}}, \dfrac{3\pi}{2}\right),\\
\left(\sqrt{\dfrac{6hN_{22}}{\delta_7}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(b+3d)}{b-c+3d-3e}}, \dfrac{\pi}{2}\right),\\
\left(\sqrt{\dfrac{6hN_{22}}{\delta_7}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2h(b+3d)}{b-c+3d-3e}}, \dfrac{3\pi}{2}\right).
\end{array} \tag{87}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \quad & \frac{9h^4(b+3d)N_{22}N_{23}}{16(b-c+3d-3e)^4(3a+b+c+3e)^3}(2bc-b^2-c^2+9ad\\
& +6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(b^2ce\\
& -bc^2d+bc^2e-3acde-c^2de+3abe^2+3bce^2-3cde^2-b^2cf\\
& -3bcdf+b^2ef+3bdef).
\end{aligned}
$$

Assuming that $(2bc-b^2-c^2+9ad+6cd-18ae-6be-6ce-9e^2+9af+6bf+9df)(b^2ce-bc^2d+bc^2e-3acde-c^2de+3abe^2+3bce^2-3cde^2-b^2cf-3bcdf+b^2ef+3bdef) \neq 0$ and (86) hold. This assumption is satisfied for the value $a = 7, b = -3, c = -2, d = -5, f = -2, e = -1, h = 1$. Then $J_{f_1(S^*)} \neq 0$ and the four zeros (87) of system (17) provide four periodic solutions of differential system (15).

If $c - b - 3d + 3e \neq 0$, $3a + 2b + 3d = 0$ and $3a + b + c + 3e = 0$ we obtain $f_{14} = 3hN_{22}/(4(b-c+3d-3e))$=constant.

If $c - b - 3d + 3e \neq 0$ and $3a + 2b + 3d \neq 0$, solving $f_{12} = 0$ we have from $f_{12} = 0$ that $R = \sqrt{\dfrac{2bh + 6dh + 3ar^2 + 2br^2 + 3dr^2}{b-c+3d-3e}}$, substituting R in f_{14} and solving $f_{14} = 0$ if

$$\omega_3 = b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce + 9e^2 - 9af - 6bf - 9df \neq 0,$$

so $r = \sqrt{\dfrac{6hN_{24}}{\omega_3}}$, $R = \sqrt{\dfrac{2hN_{25}}{\omega_3}}$ and $\rho = \sqrt{\dfrac{2hN_{26}}{\omega_3}}$ where $N_{24} = cd - be - ce - 3e^2 + bf + 3df$, $N_{25} = b^2 - bc - 9ad - 3cd + 9ae + 3be$ and $N_{26} = c^2 - bc + 9ae + 3ce - 9af - 3bf$.

With the condition that

$$
\begin{aligned}
& \omega_3 = b^2 - 2bc + c^2 - 9ad - 6cd + 18ae + 6be + 6ce\\
& \qquad +9e^2 - 9af - 6bf - 9df \neq 0,\\
& c - b - 3d + 3e \neq 0, \quad 3a + 2b + 3d \neq 0,\\
& hN_{24}\omega_3 > 0, \quad hN_{25}\omega_3 > 0 \quad \text{and} \quad hN_{26}\omega_3 > 0,
\end{aligned} \tag{88}
$$

system (17) for $q = 1$ and $q = 3$ has four zeros $S^* = (r, \alpha, R, \beta)$ with $\rho = \sqrt{\dfrac{2hN_{26}}{\omega_3}}$ given by

$$
(89) \qquad
\begin{array}{c}
\left(\sqrt{\dfrac{6hN_{24}}{\omega_3}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{25}}{\omega_3}}, \dfrac{\pi}{2}\right), \\
\left(\sqrt{\dfrac{6hN_{24}}{\omega_3}}, \dfrac{\pi}{2}, \sqrt{\dfrac{2hN_{25}}{\omega_3}}, \dfrac{3\pi}{2}\right), \\
\left(\sqrt{\dfrac{6hN_{24}}{\omega_3}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{25}}{\omega_3}}, \dfrac{\pi}{2}\right), \\
\left(\sqrt{\dfrac{6hN_{24}}{\omega_3}}, \dfrac{3\pi}{2}, \sqrt{\dfrac{2hN_{25}}{\omega_3}}, \dfrac{3\pi}{2}\right).
\end{array}
$$

The Jacobian is

$$
\begin{aligned}
J_{f_1(s*)} = \quad & \frac{9h^4 N_{24} N_{25} N_{26}}{16\omega_3^3}(bc^2 d - b^2 ce - bc^2 e + 3acde + c^2 de - 3abe^2 \\
& -3ace^2 - 5bce^2 + b^2 cf + 3bcdf + 3abef + b^2 ef).
\end{aligned}
$$

Assuming $(bc^2 d - b^2 ce - bc^2 e + 3acde + c^2 de - 3abe^2 - 3ace^2 - 5bce^2 + b^2 cf + 3bcdf + 3abef + b^2 ef) \neq 0$ and (88) hold. This supposition is not empty because the value $a = 5, b = -9, c = -8, d = -31, f = -2, e = -1, h = -1$ satisfy it. Therefore $J_{f_1(S^*)} \neq 0$ and the four zeros (89) of system (17) provide four periodic solutions of differential system (15).

If $c - b - 3d + 3e \neq 0$, $3a + 2b + 3d \neq 0$ and $\omega_3 = 0$ we have $f_{14} = -3hN_{24}/(4(b - c + 3d - 3e))$ =constant.

Proof of Lemma 2. Following the averaging theory, $(r^*, \alpha^*, R^*, \beta^*)$ is a periodic solution of (15) means that

$$
(90) \qquad
\begin{aligned}
r(t,\varepsilon) = & \quad r^* + O(\varepsilon), \\
\alpha(t,\varepsilon) = & \quad \alpha^* + O(\varepsilon), \\
R(t,\varepsilon) = & \quad R^* + O(\varepsilon), \\
\beta(0,\varepsilon) = & \quad \beta^* + O(\varepsilon).
\end{aligned}
$$

Adding the fact that $\rho = \sqrt{2h - r^{*2} - R^{*2}}$ so system (90) becomes

$$
(91) \qquad
\begin{aligned}
r(t,\varepsilon) = & \quad r^* + O(\varepsilon), \\
\rho(t,\varepsilon) = & \quad \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\
\alpha(t,\varepsilon) = & \quad \alpha^* + O(\varepsilon), \\
R(t,\varepsilon) = & \quad R^* + O(\varepsilon), \\
\beta(t,\varepsilon) = & \quad \beta^* + O(\varepsilon).
\end{aligned}
$$

We reconsider the variable,t, the temps instead of θ and the (91) becomes

$$\begin{aligned} r(t,\varepsilon) = &\quad r^* + O(\varepsilon), \\ \theta(t,\varepsilon) = &\quad t + O(\varepsilon), \\ \rho(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}} + O(\varepsilon), \\ \alpha(t,\varepsilon) = &\quad \alpha^* + O(\varepsilon), \\ R(t,\varepsilon) = &\quad R^* + O(\varepsilon), \\ \beta(t,\varepsilon) = &\quad \beta^* + O(\varepsilon). \end{aligned} \tag{92}$$

Now using the change of variables (10), the system (92) becomes

$$\begin{aligned} x(t,\varepsilon) = &\quad r^* \cos t + O(\varepsilon^{3/2}), \\ y(t,\varepsilon) = &\quad \sqrt{2h - r^{*2} - R^{*2}} \cos(\alpha^* - t) + O(\varepsilon^{3/2}), \\ z(t,\varepsilon) = &\quad R^* \cos(\beta^* - t) + O(\varepsilon^{3/2}), \\ p_x(t,\varepsilon) = &\quad r^* \sin t + O(\varepsilon^{3/2}), \\ p_y(t,\varepsilon) = &\quad \sqrt{2h + r^{*2} - R^{*2}} \sin(\alpha^* - t) + O(\varepsilon^{3/2}), \\ p_z(t,\varepsilon) = &\quad R^* \sin(\beta^* - t) + O(\varepsilon^{3/2}). \end{aligned} \tag{93}$$

Finally we reused the scaling $x = \sqrt{\varepsilon}\, X$, $y = \sqrt{\varepsilon}\, Y$,$z = \sqrt{\varepsilon}\, Z$, $p_x = \sqrt{\varepsilon}\, p_X$, $p_y = \sqrt{\varepsilon}\, p_Y$ and $p_z = \sqrt{\varepsilon}\, p_Z$ and (93) becomes

$$\begin{aligned} x(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad r^* \cos t + O(\varepsilon^{3/2}), \\ y(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad \sqrt{2h + r^{*2} - R^{*2}} \cos(\alpha^* - t) + O(\varepsilon^{3/2}), \\ z(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad R^* \cos(\beta^* - t) + O(\varepsilon^{3/2}), \\ p_x(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad r^* \sin t + O(\varepsilon^{3/2}), \\ p_y(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad \sqrt{2h + r^{*2} - R^{*2}} \sin(\alpha^* - t) + O(\varepsilon^{3/2}), \\ p_z(t,\varepsilon) = &\quad \sqrt{\varepsilon} \quad R^* \sin(\beta^* - t) + O(\varepsilon^{3/2}). \end{aligned}$$

□

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References

[1] Abraham R. and Marsden J.E., *Foundations of Mechanics*, Benjamin, Reading, Masachusets, 1978.

[2] Belomte, C. Bocaaletti, D., Pucacco, G., *On the orbit structure of the logarithm potential*, Astrophys. J.**669**, 202–217 (2007).

[3] Calzeta, E., Hasi, C.E., *Chaotic Friedmann-Robertson-Walker cosmology*, Class. Quantum Gravity. **10**, 1825–1841 (1993).

[4] Hawkings, S.W, *Arrow of time in cosmology*, Phys. Rev. D **32**, 2489–2495 (1985).

[5] Lembarki, F., Llibre, J., *Periodic orbits for the generalized Yang-Mills Hamiltonian system in dimension* 6, Nonlinear Dyn. **76**, 1807–1819 (2014).

[6] Llibre, J., Makhlouf, A., *Periodic orbits of the generalized Friedmann-Robertson-Walker Hamiltonian systems*, Astrophys. **344**, 46–50 (2013).

[7] Merrit, D., Valluri, M., *Chaos and mixing in triaxial stellar systems*, Astrophys. J **471**, 82–105 (1996).

[8] Page, D., *Will entroy decrease if the universe recollapses?*, Phys. Rev. D **32**, 2496–2499 (1991).

[9] Papaphilippou Y., Laskar J., *Frequency map analysis and global dynamics in a galactic potential with two degrees of freedom*, Astron. Astrophys. **307**, 427–449 (1996).

[10] Papaphilippou Y., Laskar J., *Global dynamics of triaxial galactic models though frequency analysis*, Astron. Astrophys. **329**, 451–481 (1998).

[11] Pucacco G., Boccaletti, D., Belmonte, C. *Quantitative predictions with detuned normal forms*, Celest. Mech. Dyn. Astron. **102**, 163–176 (2008).

[12] Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, Universitext Springer Verlag, 1996.

[13] Zhao, H.S, Carollo, C.M., De Zeeuw, T., *Can galactic nuclei be non-axisymmetric? The parameter space of power-law discs.*, Mon. Not. R. Astron. Soc.**304**, 457–464 (1999).

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