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MELNIKOV FUNCTIONS FOR PERIOD ANNULUS, NONDEGENERATE CENTERS, HETEROCLINIC AND HOMOCLINIC CYCLES

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We give sufficient conditions in terms of the Melnikov functions in order that an analytic or a polynomial differential system in the real plane has a period annulus.

We study the first nonzero Melnikov function of the analytic differential systems in the real plane obtained by perturbing a Hamiltonian system having either a nondegenerate center, a heteroclinic cycle, a homoclinic cycle, or three cycles obtained connecting the four separatrices of a saddle. All the singular points of these cycles are hyperbolic saddles.

Finally, using the first nonzero Melnikov function we give a new proof of a result of Roussarie on the finite cyclicity of the homoclinic orbit of the integrable system when we perturb it inside the class of analytic differential systems.

1. Introduction and statement of the main results.

We consider the planar vector fields $\mathcal{X}_\epsilon$ associated to the system:

$$\begin{aligned} \dot{x} &= X(x,y,\lambda,\epsilon) = p(x,y) + \epsilon P(x,y,\lambda,\epsilon), \\ \dot{y} &= Y(x,y,\lambda,\epsilon) = q(x,y) + \epsilon Q(x,y,\lambda,\epsilon), \end{aligned} \tag{1}$$

where X, Y depend analytically on their variables and parameters $\lambda \in \Lambda$, and $\epsilon \in \mathbf{R}$, $\Lambda \subset \mathbf{R}^r$ is an open region. Assume that for $\epsilon = 0$, system (1) has a period annulus; i.e., a continuous family of periodic orbits. As usual, the dot denotes derivative with respect to the time variable t. We say that system (1) with $\epsilon = 0$ is the *unperturbed* system, while system (1) with $\epsilon \neq 0$ is the *perturbed* one.

Given any compact subset $\mathbf{D}$ of Λ and $\epsilon_0 > 0$ small, we assume that there is a transversal section J to the vector fields $\mathcal{X}_\epsilon$ in the region covered by the period annulus for $|\epsilon| < \epsilon_0$ and $\lambda \in \mathbf{D}$. Let u be an analytical parameterization of J. Then there is a subsection $\Sigma \subset J$ such that the Poincaré return map $(u,\lambda,\epsilon) \mapsto \Pi(u,\lambda,\epsilon)$ is defined from $\Sigma \times \mathbf{D} \times (-\epsilon_0, \epsilon_0)$ to J. The *displacement function* $d(u,\lambda,\epsilon)$ is defined as $d(u,\lambda,\epsilon) = \Pi(u,\lambda,\epsilon) - u$. Since system (1) has a period annulus for $\epsilon = 0$, we have $d(u,\lambda,0) \equiv 0$,