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J. Differential Equations 259 (2015) 6494–6509

Journal of Differential Equations

www.elsevier.com/locate/jde

# Parallelization of the Lyapunov constants and cyclicity for centers of planar polynomial vector fields

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Received 11 July 2015; revised 26 July 2015

Available online 5 August 2015

## Abstract

Christopher in 2006 proved that under some assumptions the linear parts of the Lyapunov constants with respect to the parameters give the cyclicity of an elementary center. This paper is devoted to establish a new approach, namely parallelization, to compute the linear parts of the Lyapunov constants. More concretely, it is shown that parallelization computes these linear parts in a shorter quantity of time than other traditional mechanisms.

To show the power of this approach, we study the cyclicity of the holomorphic center $\dot{z} = iz + z^2 + z^3 + \cdots + z^n$ under general polynomial perturbations of degree $n$, for $n \leq 13$. We also exhibit that, from the point of view of computation, among the Hamiltonian, time-reversible, and Darboux centers, the holomorphic center is the best candidate to obtain high cyclicity examples of any degree. For $n = 4, 5, \ldots, 13$, we prove that the cyclicity of the holomorphic center is at least $n^2 + n - 2$. This result gives the highest lower bound for $M(6), M(7), \ldots, M(13)$ among the existing results, where $M(n)$ is the maximum number of limit cycles bifurcating from an elementary monodromic singularity of polynomial systems of degree $n$. As a direct corollary we also obtain the highest lower bound for the Hilbert numbers $H(6) \geq 40$, $H(8) \geq 70$, and $H(10) \geq 108$, because until now the best result was $H(6) \geq 39$, $H(8) \geq 67$, and $H(10) \geq 100$.


*MSC:* 34C23; 34C25; 34C07

*Keywords:* Parallelization; Center cyclicity; Planar polynomial vector field; Lyapunov constant

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http://dx.doi.org/10.1016/j.jde.2015.07.027
0022-0396/

## 1. Introduction

Poincaré began investigating limit cycles of planar polynomial differential systems in the 1880s. In 1900 David Hilbert presented a list of 23 problems in the International Congress of Mathematicians in Paris. The second part of the 16th problem is the estimation of the maximal number, $H(n)$, and relative positions of limit cycles for planar polynomial vector fields of degree $n$. By now, over a century, within the framework of investigation of this problem, numerous theoretical and numerical results were obtained, see the survey articles of Ilyashenko and Li, [11,13]. However, this problem remains almost completely unsolved even for quadratic vector fields. The best lower bounds of $H(n)$ for different $n$ among the literature can be found in [9] except that $H(4) \geq 26$ and $H(6) \geq 39$ which was obtained by [12] and [14] respectively.

There are several particular versions of Hilbert's 16th problem. Arnold in 1977 proposed a so-called weakened version to study the number of isolated zeroes of the Abelian integrals [1]. This number gives the number of limit cycles bifurcating from the period annulus of Hamiltonian systems. The second version is to determine the maximal number of limit cycles of Liénard systems, see [10] and the references therein. Another particular version of Hilbert's 16th problem is to estimate the maximum number $M(n)$ of small amplitude limit cycles bifurcating from an elementary center or an elementary focus, see [21].

The answer to the question about which is the value of $M(n)$ for any $n$ is only known for degree 2. Bautin [2] proved that $M(2) = 3$. For cubic system without quadratic terms, Sibirskiĭ in [18] proved that at most five limit cycles could be bifurcated from one critical point. Żoładek in [21] found an example where 11 limit cycles could be bifurcated from the center of a cubic system. Christopher in [4] provided a simple proof of Żoładek's result. Bondar and Sadovskiĭ in [3] also provide an example of a family of cubic systems which have at least 11 limit cycles. Recently, Żoładek revisited his example in [23]. Perturbing concrete examples of quartic and quintic systems, Giné in [8] proved that $M(4) \geq 21$ and $M(5) \geq 26$. Very recently, Liang and Torregrosa in [14] proved that $M(6) \geq 39$, $M(7) \geq 34$, and $M(8) \geq 46$.

An efficient method to produce limit cycles from a singularity of the center–focus type is calculating the Lyapunov quantities when small perturbations are considered, see [2,4,8] and the references therein. There are several ways to introduce the Lyapunov quantities. The reader can find a suitable definition in many standard textbooks of ordinary differential equations, see for example [5]. In what follows we will briefly introduce some definition, notation and symbols which are very closely to the main topic of the present paper.

It is well known that a planar polynomial differential system of degree $n$ which has an elementary center or weak focus at the origin can be written using complex coordinates, $z = x + iy$, in the form

$$\dot{z} = iz + p_n(z, \bar{z}, \boldsymbol{\lambda}), \tag{1}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$ and $p_n(z, \bar{z}, \boldsymbol{\lambda}) = \lambda_1 z + \sum_{k+\ell=2}^{n} c_{k,\ell}(\lambda_2, \ldots, \lambda_m) z^k \bar{z}^\ell$ with $c_{k,\ell}(\lambda_2, \ldots, \lambda_m) \in \mathbb{C}$.

There is always an analytic positive definite function $V(z, \bar{z})$ in a neighborhood of the origin such that $X(V) = \sum_{k=0}^{\infty} v_k (z\bar{z})^{k+1}$, where $X$ is the vector field associated to equation (1). That is, $X(V)$ is the rate of change of $V$ along the orbits of (1). The coefficient $v_k = v_k(\boldsymbol{\lambda})$ is called the $k$-Lyapunov constant of (1) at the origin. Obviously, $v_0 = 2\lambda_1$. The origin is a center if and only if $v_k = 0$ for all $k \geq 0$. We call the set of parameters for which all the Lyapunov constants vanish the *center variety*. If $\lambda_1 \neq 0$, then equation (1) has a strong focus at the origin. If $v_k$ $(k \geq 1)$ is the first nonzero constant, then the origin is a weak focus of order $k$. The focus order $k$ is the upper bound of the numbers of limit cycles which bifurcate from the focus under analytic perturbations, see [15].

In case that $\lambda_1 = 0$, $v_k$ $(k = 1, 2, \dots)$ are polynomials in $c_{k,\ell}(\boldsymbol{\lambda})$. Moreover, from [17] we know that $v_k$ is determined modulo $v_0, \dots, v_{k-1}$ in spite of the choice of function $V$ is not unique. Let $L(0) = 2\pi\lambda_1$ and $L(k) = v_k$ modulo $v_0, \dots, v_{k-1}$ for $k = 1, 2, \dots$. We call $L(k)$ the $k$-Lyapunov quantities. According to the Hilbert Basis Theorem, the ideal generated by the Lyapunov quantities has a finite number of generators. Thus, theoretically to distinguish a center from a focus, or to determine the order of a weak focus, can be solved in finite number of steps. However, for any given case, with few exceptions, it is unknown a priori how many steps are required.

In general the calculation of the Lyapunov constants or Lyapunov quantities by hand is impossible except in the simplest cases. Therefore, the research on the computation of the Lyapunov constants using computers is attracting more and more attention. Several computational methods have been developed, see the paper [20] and the textbook [5] for instance.

However, even when the computation is implemented in a computer, the computational problems are still very hard due to the big size, besides the number, of the coefficients of the polynomials in the parameters of the vector field. Christopher in [4] developed a simple computational approach to estimate lower bounds for the cyclicity of centers. The idea consists in taking into account the lowest terms of the Lyapunov quantities with respect to the parameters. As it was shown by several examples, this approach has three nice aspects. The first is the removal of the necessity of lengthy calculations. The second is that the complex independence arguments are replaced by the linear independence ones. The third is that it gives room for a more creative approach to estimating cyclicity.

The idea behind Christopher's approach is the following. Suppose that $\mathbf{s}$ is a point on the center variety. If we can choose independent $L(0), L(1), \dots, L(k)$ values in a neighborhood of $\mathbf{s} \in \mathbb{C}^m$, then with a properly choice of the parameter values we can have that $0 < |L(0)| \ll |L(1)| \ll |L(2)| \ll \cdots \ll |L(k)|$ and $L(0), L(1), L(2), \dots, L(k)$ having alternate signs. Hence according to [16], we can produce $k$ limit cycles one by one.

Christopher has proved that in many cases it suffices to calculate the linear part of $L(i)$ with respect to the parameters. See the following theorem.

**Theorem 1.1.** *(See [4].) Suppose that* $\mathbf{s}$ *is a point on the center variety and that the first $k$ of the $L(i)$ have independent linear parts (with respect to the expansion of $L(i)$ about* $\mathbf{s}$*), then* $\mathbf{s}$ *lies on a component of the center variety of codimension at least $k+1$ and there are bifurcations which produce $k$ limit cycles locally from the center corresponding to the parameter value* $\mathbf{s}$.

Applying this Theorem, Christopher in [4] studies the cubic center $C_{31}$ in Żołądek's classification [21]. A computation shows that the linear parts of $L(1), L(2), \dots, L(11)$ are independent in the parameters. Since $L(0) = 2\pi\lambda_1$ already, 11 limit cycles can bifurcate from this center. This number of limit cycles coincides with the one obtained in [21]. Giné in [8] used Christopher's method (together with the second order bifurcations) to obtain high cyclicities for several classes of centers.

However, for polynomial systems of high degree with a lot of parameters, in practice the computation of the linear parts of the Lyapunov constants is still very complicated. In fact, the systems considered in both [4] and [8] are of very low degree (degrees 3, 4 in [4] and degree 5 in [8]). While Giné studies the polynomial systems of degrees 6, 8, 9, he has to assume that the system has only homogeneous nonlinear terms [8]. We observe that, using a computer to do the calculations for general polynomial systems of high degree, we need to wait a long time. For example, for the holomorphic center $\dot{z} = iz + z^2 + z^3 + \cdots + z^9$ under the general polynomial perturbation of degree 9 without linear nor constant terms (104 real parameters) our computer needs more than 4 days of CPU time.[1]

Therefore, in order to go further with Christopher's method, we need to reduce the computation waiting time. The main aim of the present paper is to develop a parallelization procedure for computing the linear parts of the Lyapunov constants. We define $M_l(n)$ as the number of limit cycles which bifurcate from the polynomial elementary monodromic singularity of degree $n$, by using linear parts of the Lyapunov constants with respect to the parameters. We call $M_l(n)$ the *linear cyclicity* for polynomial system of degree $n$. It is clear that $M_l(n) \leq M(n)$. Thus $M_l(n)$ give a lower bound of $M(n)$. For any given polynomial system of degree $n$, if we perturb its coefficients, starting from the quadratic terms, essentially there will be at most $n^2 + 3n - 4$ free parameters and hence $M_l(n) \leq n^2 + 3n - 4$. We would like to mention here that the author of [7] conjectured this upper bound should be less than or equal to $n^2 + 3n - 7$.

Another work of this paper is to find a highest possible lower bound for $M_l(n)$ (and hence for $M(n)$), for several relative large degree $n$. Employing the parallelized procedure we are able to deal with the cases $n = 7, 8, \ldots, 13$. However, a priori we do not know which type of centers will give rise to the highest cyclicity. Thus we will search for the best candidate among the Hamiltonian centers, the time-reversible centers and some Darboux centers. After calculating about 6000 examples we have found that the best candidate to provide the highest cyclicity are the Darboux ones. Nevertheless, in general the Darboux centers are very hard to find and hence, instead we consider a class of holomorphic centers with the complex form $\dot{z} = iz + \sum_{j=2}^{n} z^j$. This class of centers is also a family of Darboux centers because the equation $\dot{z} = f(z)$ possesses an integrating factor of the form $(f(z)\overline{f(z)})^{-1}$, see [6]. Our result is the following theorem.

**Theorem 1.2.** *The cyclicity of the holomorphic center $\dot{z} = iz + \sum_{j=2}^{n} z^j$ is at least $n^2 + n - 2$ for $4 \leq n \leq 13$ and 9 for $n = 3$, under general polynomial perturbations of degree $n$.*

As a direct consequence of Theorem 1.2 we obtain the highest lower bounds of $M(n)$ for $5 \leq n \leq 13$ among the literature.

**Corollary 1.3.** *For each $n = 5, 6, \ldots, 13$ we have $M(n) \geq n^2 + n - 2$.*

Since $H(n) \geq M(n)$, we can also improve the Hilbert number when $n = 6, 8, 10$. The best previous known results show that $H(6) \geq 39$, $H(8) \geq 67$, and $H(10) \geq 100$. Now we get from the above result that:

**Corollary 1.4.** *Suppose that $H(n)$ is the Hilbert number for the class of polynomial vector fields of degree $n$. Then $H(6) \geq 40$, $H(8) \geq 70$, and $H(10) \geq 108$.*

In the proof of Theorem 1.2 we will show how the parallelization procedure reduces the waiting time of computation, see Table 1 in Section 4. From this table we also found that it is almost impossible to compute the cases of degree 11, 12, and 13 in the traditional way.

[1] The computations are done with MAPLE 18 in a Xeon computer (CPU E5-450, 3.0 GHz, RAM 32 Gb) with GNU Linux.

This paper is organized as follows. In Section 2 we establish the linear property for the linear parts of the Lyapunov quantities with respect to the parameters. This property is the foundation of the parallelization procedure. An example is also provided to illustrate how it works. In Section 3 we exhibit that the best center candidate to produce high cyclicity is a Darboux center. In Section 4 we study the number of limit cycles bifurcating from some holomorphic centers, using the parallelization procedure for computing the linear parts of the Lyapunov constants.

## 2. Linear property of the linear parts of the Lyapunov constants

This section is devoted to the proof of the following theorem which shows the linear property of the linear parts of the Lyapunov constants. Afterward, we give an example to illustrate how this property can be applied in the explicit computations.

**Theorem 2.1.** *Let $p(z,\bar z)$ be a polynomial starting with terms of degree 2. Let $Q_j(z,\bar z,\boldsymbol\lambda)$ be analytic functions such that $Q_j(0,0,\boldsymbol\lambda)\equiv 0$ and $Q_j(z,\bar z,\mathbf{0})\equiv 0$, for $j=1,\dots,s$. Let $a_1,\dots,a_s$ be any $s$ fixed constants. Suppose that $v_k^{Q_j}$ are the $k$-Lyapunov constants of equations*

$$\dot z = iz + p(z,\bar z) + Q_j(z,\bar z,\boldsymbol\lambda),\ \boldsymbol\lambda\in\mathbb{C}^m,\ \ \textit{for } j=1,\dots,s. \tag{2}$$

*Then the linear part, with respect to the components of $\boldsymbol\lambda$, of $a_1 v_k^{Q_1}+\dots+a_s v_k^{Q_s}$ is the linear part of the $k$-Lyapunov constant of equation*

$$\dot z = iz + p(z,\bar z) + a_1Q_1(z,\bar z,\boldsymbol\lambda)+\dots+a_sQ_s(z,\bar z,\boldsymbol\lambda), \tag{3}$$

*with respect to the components of $\boldsymbol\lambda$.*

**Proof.** We recall that the way to find the Lyapunov constants of a given vector field $X = X(z,\bar z,\boldsymbol\lambda)$ at the origin is, as it is given in [4], as follows. We seek a positive definite analytic function $W=W(z,\bar z,\boldsymbol\lambda)$ in a neighborhood of the origin such that

$$X(W)=\sum_{k=0}^{\infty}\beta_k r^{2k+2},$$

with $r^2=z\bar z$. If we want to find the term of degree $j$ of $\beta_k$ with respect to the parameters $\boldsymbol\lambda$, it turns out to be equivalent to solve the following equations step by step:

$$X_0(W_0)=\begin{cases}0, & \text{if } X_0 \text{ has a center at the origin,}\\ \bar\beta_\ell r^{2\ell+2}+\cdots, & \text{if } X_0 \text{ has a weak focus at the origin with order } \ell,\end{cases} \tag{4}$$

$$X_0(W_1)+X_1(W_0)=\sum_{k=0}^{\infty}\beta_{k,1}r^{2k+2},$$

$$X_0(W_j)+X_1(W_{j-1})+\dots+X_j(W_0)=\sum_{k=0}^{\infty}\beta_{k,j}r^{2k+2}, \tag{5}$$

where $X_j$, $W_j$, and $\beta_{k,j}$ are the terms of degree $j$ of $X$, $W$, and $\beta_k$, respectively, with respect to the parameters $\boldsymbol{\lambda}$.

This means that $\beta_{k,1}$ is the linear part of the $k$-Lyapunov constant of vector field $X$ if and only if there exists a positive definite analytic function $W_0 = W_0(z,\bar{z})$ and $W_1 = W_1(z,\bar{z},\boldsymbol{\lambda})$ (linear in $\boldsymbol{\lambda}$) such that both (4) and (5) are satisfied.

Suppose that $W^{Q_j}(z,\bar{z},\boldsymbol{\lambda})$ is the positive definite analytic function in a neighborhood of the origin such that

$$X^{Q_j}(W^{Q_j}) = \sum_{k=0}^{\infty} v_k^{Q_j} r^{2k+2}, \tag{6}$$

where $X^{Q_j}$ are the vector fields associated to equations (2).

Let

$$X^{Q_j} = X_0 + X_1^{Q_j} + X_2^{Q_j} + \cdots, \quad W^{Q_j} = W_0 + W_1^{Q_j} + W_2^{Q_j} + \cdots,$$

where $X_k^{Q_j}$ (resp. $W_k^{Q_j}$) is the term of degree $k$ of $X^{Q_j}$ (resp. $W^{Q_j}$) with respect to $\boldsymbol{\lambda}$. Denoted by $v_{k,1}^{Q_j}$ the linear term of $v_k^{Q_j}$ with respect to the parameters. It follows from (6) that

$$X_0(W_1^{Q_j}) + X_1^{Q_j}(W_0) = \sum_{k=0}^{\infty} v_{k,1}^{Q_j} r^{2k+2}, \quad j = 1, \ldots, s. \tag{7}$$

By the linear property of vector field, we obtain from (7) that

$$X_0\left(\sum_{j=1}^{s} a_j W_1^{Q_j}\right) + \left(\sum_{j=1}^{s} a_j X_1^{Q_j}\right)(W_0) = \sum_{k=0}^{\infty} \left(a_1 v_{k,1}^{Q_1} + \cdots + a_s v_{k,1}^{Q_s}\right) r^{2k+2}.$$

Therefore, there exists a function $W_1 := a_1 W_1^{Q_1} + \cdots + a_s W_1^{Q_s}$ being linear in the parameters such that

$$X_0(W_1) + \left(a_1 X_1^{Q_1} + \cdots + a_s X_1^{Q_s}\right)(W_0) = \sum_{k=0}^{\infty} \left(a_1 v_{k,1}^{Q_1} + \cdots + a_s v_{k,1}^{Q_s}\right) r^{2k+2}.$$

We remark that $a_1 X_1^{Q_1} + \cdots + a_s X_1^{Q_s}$ is exactly the linear part of the vector field $X$ associated to equation (3), thus $W_1$ and $\beta_{k,1} := a_1 v_{k,1}^{Q_1} + \cdots + a_s v_{k,1}^{Q_s}$ are the solutions of equation (5). Clearly equation (4) is satisfied automatically and hence the proof is complete. □

The advantage of applying Theorem 2.1 is the following. In computation of the linear part of the Lyapunov constants with respect to the parameters of polynomial equation $\dot{z} = iz + p_n(z,\bar{z}) + \sum_{k+\ell=2}^{n} \lambda_{k,\ell} z^k \bar{z}^\ell$, we will one-by-one compute the linear part of each Lyapunov constant of equation $\dot{z} = iz + p_n(z,\bar{z}) + \lambda_{k,\ell} z^k \bar{z}^\ell$. After all the computations have been done, we take the summation of all the results and hence we obtain the linear part of the Lyapunov constant of the original complete equation. The computation done in this way is called *parallelization*. Additionally, when we implement this procedure, the computations in each separate equation given by each perturbation monomial are shorter in size and time. Clearly, if we use several computers to do the computations for different $\lambda_{k,\ell} z^k \bar{z}^\ell$ simultaneously, then the total waiting time will be reduced drastically. We will compare, at the end of Section 4, both the time differences between the computations when they are done, when it is possible, in the traditional way using only one computer and with the parallelization procedure using a cluster of computers.

Next we give an example to illustrate the parallelization procedure. Consider the equation

$$\dot{z} = iz + 10z^2 + 5z\bar{z} + (3+4i)\bar{z}^2 + (\lambda_2 i z^2 + \lambda_3 z\bar{z} + \lambda_4 \bar{z}^2), \tag{8}$$

where $\lambda_j$ $(j = 2, 3, 4)$ are real small parameters.

The unperturbed system of (8) in its real form is

$$\dot{x} = -y + 18x^2 + 8xy - 8y^2, \quad \dot{y} = x + 4x^2 + 14xy - 4y^2. \tag{9}$$

It has a first integral

$$H = \frac{(80x^3 - 480x^2y + 960xy^2 - 640y^3 + 120xy - 240y^2 - 30y - 1)^2}{(20x^2 - 80xy + 80y^2 + 20y + 1)^3}.$$

Therefore, system (9) has a center at the origin.

Now we employ the parallelization to compute the linear part of the Lyapunov constants for equation (8) at the origin. By direct computation using the algorithm of [5] we obtain the first three linear parts of the Lyapunov constants of equations $(8)_{\lambda_3=\lambda_4=0}$, $(8)_{\lambda_2=\lambda_4=0}$, and $(8)_{\lambda_2=\lambda_3=0}$, which are respectively

$$v_1^{\ell,Q_1} = -10\pi\lambda_2, \ v_2^{\ell,Q_1} = 16\,000\pi\lambda_2, \ v_3^{\ell,Q_1} = -\frac{682\,934\,375\pi\lambda_2}{18},$$
$$v_1^{\ell,Q_2} = 0, \ v_2^{\ell,Q_2} = \frac{2000\pi\lambda_3}{3}, \ v_3^{\ell,Q_2} = -\frac{16\,356\,250\pi\lambda_3}{9},$$

and

$$v_1^{\ell,Q_3} = 0, \ v_2^{\ell,Q_3} = 0, \ v_3^{\ell,Q_3} = 18\,750\pi\lambda_4.$$

According to Theorem 2.1, the first three linear parts of the Lyapunov constants of equation (8) are

$$\begin{aligned}
v_1^\ell &= v_1^{\ell,Q_1} + v_1^{\ell,Q_2} + v_1^{\ell,Q_3} = -10\pi\lambda_2,\\
v_2^\ell &= v_2^{\ell,Q_1} + v_2^{\ell,Q_2} + v_2^{\ell,Q_3} = 16\,000\pi\lambda_2 + \frac{2000\pi\lambda_3}{3},\\
v_3^\ell &= v_3^{\ell,Q_1} + v_3^{\ell,Q_2} + v_3^{\ell,Q_3} = -\frac{682\,934\,375\pi\lambda_2}{18} - \frac{16\,356\,250\pi\lambda_3}{9} + 18\,750\pi\lambda_4.
\end{aligned} \tag{10}$$

With the goal of checking the correctness of (10), we also compute the Lyapunov constants of equation (8) in a direct way. It turns out that

$$
\begin{aligned}
v_1 &= -2\pi\lambda_2(5+\lambda_3),\\
v_2 &= \frac{2\pi(5+\lambda_3)}{3}(18\lambda_2^3 + 27\lambda_2\lambda_3^2 + 3\lambda_2\lambda_3\lambda_4 + 8\lambda_2^2 + 759\lambda_2\lambda_3 - 25\lambda_2\lambda_4 + 8\lambda_3^2 + 4800\lambda_2\\
&\quad + 200\lambda_3),\\
v_3 &= -\frac{\pi(5+\lambda_3)}{18}(1944\lambda_2^5 + 4572\lambda_2^3\lambda_3^2 + 8\lambda_2^3\lambda_3\lambda_4 + 3384\lambda_2\lambda_3^4 + 1112\lambda_2\lambda_3^3\lambda_4 + 3816\lambda_2^4\\
&\quad + 150\,504\lambda_2^3\lambda_3 - 28\,592\lambda_2^3\lambda_4 + 2082\lambda_2^2\lambda_3^2 + 209\,256\lambda_2\lambda_3^3 + 15\,408\lambda_2\lambda_3^2\lambda_4 + 1242\lambda_3^4\\
&\quad + 938\,100\lambda_2^3 + 31\,300\lambda_2^2\lambda_3 - 336\lambda_2^2\lambda_4 + 4\,525\,685\lambda_2\lambda_3^2 - 35\,558\lambda_2\lambda_3\lambda_4 + 69\,240\lambda_3^3\\
&\quad - 876\lambda_3^2\lambda_4 - 732\,150\lambda_2^2 + 41\,353\,200\lambda_2\lambda_3 - 491\,150\lambda_2\lambda_4 + 1\,216\,450\lambda_3^2\\
&\quad - 24\,600\lambda_3\lambda_4 + 136\,586\,875\lambda_2 + 6\,542\,500\lambda_3 - 67\,500\lambda_4).
\end{aligned}
$$

By the above expression it is not hard to check that the linear part of the above complete expressions with respect to $\lambda_2$, $\lambda_3$, $\lambda_4$ is just the same as (10).

**Remark 2.2.** From (10) we know that $v_1^\ell$, $v_2^\ell$, $v_3^\ell$ are linearly independent. Thus if we add the linear perturbation $\lambda_1 z$ to equation (8), we obtain three limit cycles which emerge from the origin. This provides a simple example of a quadratic system with three limit cycles.

**Remark 2.3.** Usually the expressions of the Lyapunov constants can be simplified by modulo the previous ones. In fact, in our example the first three Lyapunov constants $v_1$, $v_2$, and $v_3$ can be replaced with the three quantities $L(1) = -2\lambda_2(5+\lambda_3)$, $L(2) = 16\lambda_3^3/3 + 160\lambda_3^2 + 2000\lambda_3/3$, $L(3) = 150\lambda_3^2\lambda_4 + 4500\lambda_3\lambda_4 + 18\,750\lambda_4$. The linear part of the Lyapunov constants can also be done in the same way. But, certainly, the number of independent Lyapunov constants is invariant.

## 3. Searching the highest linear cyclicity of centers by application of parallelization

In the present section we will apply the parallelization procedure to find the best centers which give rise to the highest cyclicity explicit vector fields. More precise, we will study the limit cycles bifurcating from the Hamiltonian, time-reversible and Darboux centers, by using the linear part of the Lyapunov constants with respect to the parameters.

### *3.1. Linear cyclicity of Hamiltonian centers*

We have computed more than 4000 random Hamiltonian centers with different degrees. They are: 1000 quadratic Hamiltonian systems, 1000 cubic Hamiltonian systems, 1000 quartic Hamiltonian systems, 1000 quintic Hamiltonian systems, 300 Hamiltonian systems of degree 6, and 50 Hamiltonian systems of degree 7. In almost all cases, the number of independent linear parts of the Lyapunov constants with respect to the parameters are respectively 2, 5, 9, 14, 20, and 27. It is clear that these Hamiltonian systems cannot provide a lower bound for $M_l(n)$ bigger than $(n^2+n-2)/2$. But in almost all cases we have that number of limit cycles. Using the parallelization procedure, the total computing time is, with our computers, about 30 hours.

### 3.2. Linear cyclicity of time-reversible centers

First we remember that a system has a time-reversible center if it has a center and it is invariant (for example) by the change $(x, y, t) \to (x, -y, -t)$. We have computed 2000 random time-reversible centers with different degrees. They are: 500 quadratic systems, 500 cubic systems, 250 quartic systems, 250 quintic systems, 250 systems of degree 6, and 250 systems of degree 7. We have found that, in almost all cases, the number of independent linear parts of the Lyapunov constants with respect to the parameters are respectively 2, 6, 11, 17, 24, and 32. It is clear that these time-reversible systems cannot provide a lower bound for $M_l(n)$ bigger than $(n^2 + 3n - 6)/2$. But in almost all cases we have that number of limit cycles. Using the parallelization the total computing time is about 16 hours.

### 3.3. Linear cyclicity of general Darboux centers

In this part we will first consider some of the known Darboux centers with high linear cyclicity when the degree is fixed to $n = 2, 3, 4, 5$. The selected quadratic Darboux center is equation (8). As we have done in Section 2, it holds that $M_l(2) \geq 3$. The cubic Darboux centers are the systems in [3,4]. These systems have a Darboux first integral of the form

$$H = \frac{(-42x + 7y + 1)^3 f}{(448x^2 + 336xy + 63y^2 - 44x - 12y + 1)^3(1183x^2 - 68x + 1)}$$

with $f = -10\,752x^3 - 29\,568x^2y - 17\,640xy^2 - 3024y^3 + 1600x^2 + 2760xy + 576y^2 - 74x - 57y + 1$ and

$$H = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + 5xy^3/2 + 5y^3/2 + 15xy/8 + 15y/4 + 2)^2}, \tag{11}$$

respectively. The first has a center at the origin and the second at (342/53, 140/53). Bondar and Sadovskiĭ in [3] and Christopher in [4] actually proved that $M_l(3) \geq 11$. The quartic and quintic Darboux centers selected are respectively system (6) and system (17) appearing in [8]. After computation we check that $M_l(4) \geq 16$ and $M_l(5) \geq 23$ as Giné proves in [8].

We note that, the construction of a polynomial Darboux center, whose linear cyclicity can be computed easily, is much harder than the Hamiltonian and time-reversible centers. We finish this section providing a class of a Darboux polynomial center of degree $n$ for $n \geq 4$. The construction of these centers is inspired by the cubic family of (11), which was originally done in [22].

**Proposition 3.1.** *Let $H$ be the rational function*

$$H(x, y) = \frac{(xy^2 + Ax + B)^{n+2}}{x^n(xu(y) + v(y))^2},$$

*with $A^2 + B^2 \neq 0$, $n \geq 4$,*

$$u(y) = \sum_{j=0}^{n-4} a_j y^j + \frac{n(n+2)A^2}{8} y^{n-2} + \frac{(n+2)A}{2} y^n + y^{n+2}, \tag{12}$$

*and*

$$v(y)=\sum_{j=0}^{n-3}b_jy^j+\frac{n(n+2)AB}{4}y^{n-2}+\frac{(n+2)B}{2}y^n. \tag{13}$$

*Then the system*

$$\dot{x}=P(x,y):=-H_y/M,\quad \dot{y}=Q(x,y):=H_y/M, \tag{14}$$

*with* $M=M(x,y)=(xy^2+Ax+B)^{n+1}/(x^{n+1}(xu(y)+v(y))^3)$, *is a polynomial system of degree* $n$.

**Proof.** By direct computation we obtain that

$$P(x,y)=2x(xy^2+Ax+B)(xu'(y)+v'(y))-2(n+2)x^2y(xu(y)+v(y)),$$
$$Q(x,y)=(2xy^2+2Ax-Bn)v(y)-(n+2)Bxu(y).$$

Substituting (12) and (13) into the above expressions, it turns out that

$$\begin{aligned}P(x,y)=-2x\Bigg(&\sum_{j=0}^{n-4}(n+2-j)a_jx^2y^{j+1}+\sum_{j=0}^{n-3}(n+2-j)b_jxy^{j+1}-(Ax+B)\\&\cdot\Big(\sum_{j=1}^{n-4}ja_jxy^{j-1}+\sum_{j=1}^{n-3}jb_jy^{j-1}+\frac{n(n^2-4)A^2}{8}xy^{n-3}+\frac{n(n^2-4)AB}{4}y^{n-3}\Big)\\&-\frac{n(n+2)B^2}{2}y^{n-1}\Bigg),\\Q(x,y)=-(n+2)B&\left(\sum_{j=0}^{n-4}a_jxy^j+\frac{n(n+2)A^2}{8}xy^{n-2}\right)+(2xy^2+2Ax-nB)\sum_{j=0}^{n-3}b_jy^j\\&+\frac{n(n+2)AB}{4}(2Ax-nB)y^{n-2}-\frac{n(n+2)B^2}{2}y^n.\end{aligned}$$

Obviously, $\deg P=n$ if $A\neq 0$ and $\deg Q=n$ if $B\neq 0$. □

We can construct a Darboux center of degree $n$ of the form (14). As an example, we show, in the next proposition, that under a suitable choice of the parameters, system (14) has a center at the point $(-1,2)$.

**Proposition 3.2.** *For any integer* $n\geq 4$, *there exist polynomials* $u$, $v$ *of the form* (12) *and* (13) *respectively such that system* (14) *has a center at* $(-1,2)$.

**Proof.** Let

$$s_1(y) = \sum_{j=0}^{n-4} a_j y^j, \quad s_2(y) = \sum_{j=0}^{n-3} b_j y^j.$$

By performing straightforward computations we find that

$$P(-1,2) = 0, \ Q(-1,2) = 0, \ P_x(-1,2) + Q_y(-1,2) = 0$$

hold if

$$\begin{aligned} s_1(2) &= \frac{1}{4}B(A^2 n(n^2-4)2^{n-6} - ABn(n^2-4)2^{n-5} - Bn(n+2)2^{n-2} + \Delta_1(2)), \\ s_2(2) &= 2^{n-5}n(16B - A^2(n-2) + 2ABn) + (8+2A+Bn) \\ &\quad \cdot \left(2^{n-8}A^2 n(n-2) - 2^{n-7}ABn(n-2) - 2^{n-4}Bn + \frac{\Delta_1(2))}{4(n+2)}\right), \end{aligned} \tag{15}$$

where $\Delta_1(2) = s_1'(2) - s_2'(2)$. Furthermore, if relations (15) are true, then

$$\begin{aligned} D :&= 4(P_x(-1,2)Q_y(-1,2) - P_y(-1,2)Q_x(-1,2)) \\ &= -4\big(2^n B(-16B + A^2(n-4) - 2AB(n-2))n(n^2-4) + 64Bn\Delta_1(2) \\ &\quad - 64(8+2A-2B)s_2'(2)\big)^2 + B^2 n(-2^n(n+2)(128 + n(-2(-8+A)A \\ &\quad + 4(A-4)B + A(A-2B)n)) - 64\Delta_1(2))(2^n(16B^2 + A^3(n-4) - 3A^2B(n-4) \\ &\quad + 2AB^2(n-4))n(n^2-4) + 64(-(4+A-B+4n)\Delta_1(2) + 2(4+A-B)\Delta_2(2))), \end{aligned}$$

where $\Delta_2(2) = s_1''(2) - s_2''(2)$.

There are many choices of $A$, $B$, $s_1$, and $s_2$ such that $D > 0$ and that at $(-1,2)$ neither the denominator nor the numerator of $M$ vanish, where $M$ is the rational function defined in Proposition 3.1. Consequently, system (14) has a center at $(-1,2)$.

For example, we can choose $A = -B$, $s_1''(2) = s_2''(2)$, $s_1'(2) = s_2'(2) = a$ and then we choose $s_1(2)$, $s_2(2)$ such that relations (15) are satisfied. In this situation we have $D = 9(n-4)(n^2-4)^2 n^3 2^{2n-1} B^7 + P_6(B,n)$, where $P_6(B,n)$ is a polynomial in $B$ of degree 6. Thus, if $n > 4$ and $B$ is large enough, then $D > 0$. If $n = 4$, we take $B = 3$ and we obtain $D = 65\,536(46\,656 + 216a - a^2)$. Thus $D$ is positive when $a$ is a small number. Clearly, in any case we can choose the values of $B$ and $a$ such that the point $(-1,2)$ does not lie in the curves $xy^2 + Ax + B = 0$ and $xu(y) + v(y) = 0$. This fact completes the proof. □

**Proposition 3.3.** *There exist Darboux centers of the form (14) of degrees* 4, 5, 6, 7 *with at least* 16, 26, 35, 47 *limit cycles respectively bifurcating from the center at* $(-1,2)$ *under polynomial perturbation of the same degree, using the linear parts of the Lyapunov constants.*

**Proof.** For $n = 4, 5, 6$, and 7 we take system (14) with $A = A_n$, $B = B_n$, $u = u_n$, and $v = v_n$ in Proposition 3.1 where

$$
\begin{aligned}
&A_4=-1, && u_4(y)=y^6-3y^4+3y^2+3151,\\
&B_4=4, && v_4(y)=12y^4-24y^2-3648y+10\,668,\\
&A_5=-1, && u_5(y)=y^7-\frac{7}{2}y^5+\frac{35}{8}y^3+y-\frac{6\,465\,749}{321\,132},\\
&B_5=1, && v_5(y)=\frac{7}{2}y^5-\frac{35}{4}y^3-\frac{875\,275}{91\,752}y^2+y+\frac{691\,483}{45\,876},\\
&A_6=-1, && u_6(y)=y^8-4y^6+6y^4-\frac{40\,983\,577}{162\,912},\\
&B_6=-\frac{3}{2}, && v_6(y)=-6y^6+18y^4-\frac{9\,704\,473}{61\,092}y^2+\frac{13\,333\,373}{122\,184},\\
&A_7=-1, && u_7(y)=y^9-\frac{9}{2}y^7+\frac{63}{8}y^5+\frac{97\,542\,143}{258\,048}y^2-\frac{89\,413\,631}{64\,512}y+\frac{75\,866\,111}{64\,512},\\
&B_7=1, && v_7(y)=\frac{9}{2}y^7-\frac{63}{4}y^5.
\end{aligned}
$$

After direct calculation, we obtain the required conclusion. □

We have not check values for $n>7$ because it is clear from the above proposition that when $n$ increases the number of limit cycles using the linear parts of the Lyapunov constants is less than or equal to $n^2$. The examples in the next section provide better lower bounds for $M(n)$.

As a final conclusion of this section, we remark that among the Hamiltonian, time-reversible, and Darboux systems, the best ones to produce high cyclicity centers, using the linear part of the Lyapunov constants, are the Darboux ones. However, from the point of explicit computation for any fixed degree $n$, the construction of a Darboux center is much more difficult than other families.

## 4. Cyclicity of holomorphic center under polynomial perturbation

There are two aims in this section. The first one is to give examples showing the power of the parallelization procedure developed in Section 2. The second one is to find a good lower bound for $M(n)$, for some relative large degrees.

As we have exhibited in the previous section, the best examples to produce high cyclicity are some Darboux centers. However, from the point of view of explicit computations, the Darboux centers are hard to find. Instead we will study a particular polynomial *holomorphic center* of degree $n$. We recall that an equation $\dot z=iz+f(z)$ has a holomorphic center at the origin when $f$ is a holomorphic function such that $f(0)=0$ and $\mathrm{Re}(f'(0))=0$, see [19]. According to Proposition 3.1 of [6], the holomorphic center is also a Darboux center. The holomorphic center of degree $n$ studied in this section is

$$
\dot z=iz+z^2+z^3+\cdots+z^n.
$$

We are going to deal with a general perturbation of degree $n$ without constant term. That is

$$\dot{z} = (i + \lambda_1)z + z^2 + z^3 + \cdots + z^n + \sum_{k+\ell=2}^{n} (e_{k,\ell} + f_{k,\ell} i) z^k \bar{z}^\ell, \tag{16}$$

where $\lambda_1$, $e_{k,\ell}$ and $f_{k,\ell}$ are small real parameters.

The main result of this section is:

**Theorem 4.1.** *Equation (16) has at least* 9 *and* $n^2 + n - 2$ *limit cycles emerging from the holomorphic center for* $n = 3$ *and* $4 \le n \le 13$, *respectively.*

**Proof.** First assume in (16) that $\lambda_1 = 0$. For $4 \le n \le 13$, using the parallelization procedure, we find that the coefficients matrix of the linear part of the corresponding Lyapunov constants $v_1, v_2, v_3, \ldots, v_{n^2+n-2}$, with respect to the parameters $e_{k,\ell}$, $f_{k,\ell}$, has rank $n^2 + n - 2$. Hence $v_1^\ell, v_2^\ell, \ldots, v_{n^2+n-2}^\ell$ are linearly independent. Therefore, adding the linear perturbation $\lambda_1 z$, equation (16) has $n^2 + n - 2$ limit cycles, which arise from the holomorphic center. We have not added the explicit expressions of these linear parts because of the huge size of them. The case $n = 3$ works in the same way but the number of linearly independent linear parts of the Lyapunov constants is 9. The explicit expressions of them, for this cubic family, can be found in Appendix A. □

We note that when the perturbation has only holomorphic monomials,

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + (e_{k,0} + f_{k,0} i) z^k, \tag{17}$$

the origin remains as a center. That is, all the Lyapunov constants of equation (17) are zero and the parameters $e_{k,0}$ and $f_{k,0}$ do not make any contribution in producing limit cycles. Consequently, we have that the total number of essential parameters, $2 + 3 + \cdots + n = n^2 + n - 2$, coincides with the lower bound of the cyclicity given by the above result.

As an immediate consequence of the above result, we get a lower bound for the number of small amplitude limit cycles for polynomial vector fields of degrees $4, 5, \ldots, 13$.

**Corollary 4.2.** $M(n) \ge n^2 + n - 2$ *for every degree* $4 \le n \le 13$.

**Remark 4.3.** For $5 \le n \le 13$, Corollary 4.2 provides the highest lower bound of $M(n)$, among all the known results.

The proof of Theorem 4.1 is based on the computation, with an algebraic manipulator, of the Lyapunov constants. In practice we found that when we compute them, in the traditional way, the computation time as well as the size of the constants grow very fast with the degree $n$. To reduce the waiting time, we compute only the linear part of the Lyapunov constants for equation (16) case by case independently. That is, for each case we only compute them with one-monomial perturbation:

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + e_{k,\ell} z^k \bar{z}^\ell, \quad \text{or} \quad \dot{z} = iz + z^2 + z^3 + \cdots + z^n + i f_{k,\ell} z^k \bar{z}^\ell.$$

Finally, we add all the computational results and hence the full expression of the linear part of the Lyapunov constants is obtained. Theorem 2.1 ensures the validity of this parallelization procedure.

Table 1
Computation time for the linear parts of the Lyapunov constants of equation (16). The last column shows the rank of the coefficients matrix of the linear part of the first $n^2+n-2$ Lyapunov constants.

| $n$ | Number of cases | Total time | Waiting time | Rank |
|---|---|---|---|---|
| 3 | 10 | 8 s | 8 s | 9 |
| 4 | 18 | 2 m | 6 s | 18 |
| 5 | 28 | 12 m | 30 s | 28 |
| 6 | 40 | 1.2 h | 2 m | 40 |
| 7 | 54 | 5.8 h | 7.4 m | 54 |
| 8 | 70 | 1.4 d | 1.1 h | 70 |
| 9 | 88 | 4.9 d | 3.1 h | 88 |
| 10 | 108 | 12.3 d | 6.3 h | 108 |
| 11 | 130 | 33.2 d | 0.9 d | 130 |
| 12 | 154 | 100.1 d | 2.5 d | 154 |
| 13 | 180 | 357.6 d | 8.0 d | 180 |

Table 1 illustrates the advantage of the computation method presented in this paper for the holomorphic center (16) for $n = 3, 4, \ldots, 13$. The total CPU time is 17 months but, using the parallel procedure,[2] the real waiting time is less than 12 days.

The computations to go further in $n$ have two main constraints: the first is the time needed and the second is the memory requirements. Both restrictions make it almost impossible to calculate the effective values of the Lyapunov constants for bigger values of $n$ within a reasonable time.

**Remark 4.4.** When we handle the quadratic holomorphic center, only one limit cycle appears from the linear part of the Lyapunov constants. But we can find another one using the second order terms. Moreover, the third Lyapunov quantity can be expressed as a polynomial in the first two and the third vanishes when the first two also vanish. Hence there are no more than two limit cycles emerging from the origin for this quadratic holomorphic center. For the cubic family, also using approximations of order two, no more than 9 limit cycles can be found, nor using other cubic holomorphic centers different from (16).

## Acknowledgments

This work was done when H. Liang was visiting the Department of Mathematics of Universitat Autònoma de Barcelona. He is very grateful for the support and hospitality.

The first author is supported by the NSF of China (No. 11201086, No. 11401255) and the Excellent Young Teachers Training Program for colleges and universities of Guangdong Province, China (No. Yq2013107). The second author is partially supported by the MINECO/FEDER grants MTM2008-03437, MTM2013-40998-P, and UNAB13-4E-1604; the AGAUR grant 2014 SGR568; and the European grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

## Appendix A

In this appendix we list, for equation (16) with $n = 3$, the expressions of the linear terms of $v_1, \ldots, v_{n^2}$ with respect to parameters $e_{k,\ell}$ and $f_{k,\ell}$, which are denoted by $v_1^\ell, \ldots, v_{n^2}^\ell$.

[2] The computations are done in a cluster of eight computers with 64 cores in total.

$$
\begin{aligned}
v_1^\ell &= -2f_{1,1} + 2e_{2,1},\\
v_2^\ell &= -2f_{1,2} + 12f_{1,1} - 12e_{2,1} - 2e_{1,1} - \frac{4}{3}e_{0,2},\\
v_3^\ell &= f_{2,1} + 20f_{1,2} - 105f_{1,1} - \frac{1}{2}f_{0,3} + \frac{3}{2}f_{0,2} + 104e_{2,1} + 21e_{1,1} - \frac{3}{4}e_{0,3} + \frac{41}{3}e_{0,2},\\
v_4^\ell &= -\frac{50}{3}f_{2,1} - 258f_{1,2} + \frac{12\,752}{9}f_{1,1} + \frac{41}{5}f_{0,3} - \frac{373}{15}f_{0,2} - \frac{12\,614}{9}e_{2,1} + \frac{2}{3}e_{1,2}\\
&\quad - \frac{824}{3}e_{1,1} + \frac{111}{10}e_{0,3} - \frac{2674}{15}e_{0,2},\\
v_5^\ell &= \frac{7675}{24}f_{2,1} + \frac{57\,481}{12}f_{1,2} - \frac{2\,018\,419}{72}f_{1,1} - \frac{18\,833}{120}f_{0,3} + \frac{7151}{15}f_{0,2} + \frac{499\,441}{18}e_{2,1}\\
&\quad - \frac{33}{2}e_{1,2} + \frac{40\,881}{8}e_{1,1} - \frac{24\,659}{120}e_{0,3} + \frac{1\,791\,283}{540}e_{0,2},\\
v_6^\ell &= -\frac{296\,831}{36}f_{2,1} - \frac{29\,764\,657}{240}f_{1,2} + \frac{34\,692\,256}{45}f_{1,1} + \frac{5\,097\,221}{1260}f_{0,3} - \frac{2\,581\,051}{210}f_{0,2}\\
&\quad - \frac{45\,814\,873}{60}e_{2,1} + \frac{6674}{15}e_{1,2} - \frac{19\,047\,371}{144}e_{1,1} + \frac{368\,121}{70}e_{0,3} - \frac{30\,919\,933}{360}e_{0,2},\\
v_7^\ell &= \frac{40\,837\,813}{144}f_{2,1} + \frac{1\,032\,614\,491}{240}f_{1,2} - \frac{20\,324\,018\,167}{720}f_{1,1} - \frac{748\,011\,683}{5376}f_{0,3}\\
&\quad + \frac{6\,817\,870\,105}{16\,128}f_{0,2} + \frac{5\,035\,491\,263}{180}e_{2,1} - \frac{2\,774\,227}{180}e_{1,2} + \frac{1\,651\,130\,047}{360}e_{1,1}\\
&\quad - \frac{1\,942\,410\,935}{10\,752}e_{0,3} + \frac{17\,158\,101\,913}{5760}e_{0,2},\\
v_8^\ell &= -\frac{177\,051\,675}{14}f_{2,1} - \frac{26\,100\,328\,784}{135}f_{1,2} + \frac{35\,195\,459\,317\,213}{26\,460}f_{1,1}\\
&\quad + \frac{2\,251\,596\,398\,797}{362\,880}f_{0,3} - \frac{6\,840\,775\,814\,797}{362\,880}f_{0,2} - \frac{69\,794\,246\,470\,681}{52\,920}e_{2,1}\\
&\quad + \frac{1\,485\,135\,871}{2160}e_{1,2} - \frac{778\,666\,933\,763}{3780}e_{1,1} + \frac{5\,846\,388\,826\,217}{725\,760}e_{0,3}\\
&\quad - \frac{31\,216\,718\,867\,753}{233\,280}e_{0,2},\\
v_9^\ell &= \frac{2\,065\,517\,374\,450\,619}{2\,903\,040}f_{2,1} + \frac{111\,181\,806\,164\,273\,311}{10\,160\,640}f_{1,2}\\
&\quad - \frac{1\,595\,247\,717\,563\,460\,547}{20\,321\,280}f_{1,1} - \frac{5\,067\,118\,487\,024\,053}{14\,515\,200}f_{0,3}\\
&\quad + \frac{3\,848\,676\,339\,819\,287}{3\,628\,800}f_{0,2} + \frac{49\,448\,632\,701\,419\,887}{635\,040}e_{2,1} - \frac{3\,117\,488\,162\,039}{80\,640}e_{1,2}\\
&\quad + \frac{236\,838\,515\,220\,943\,261}{20\,321\,280}e_{1,1} - \frac{939\,874\,882\,082\,747}{2\,073\,600}e_{0,3}\\
&\quad + \frac{3\,462\,158\,492\,424\,717\,361}{457\,228\,800}e_{0,2}.
\end{aligned}
$$

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