WEAK-FOCI OF HIGH ORDER AND CYCLICITY

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Abstract. A particular version of the 16th Hilbert's problem is to estimate the number, $M(n)$, of limit cycles bifurcating from a singularity of center-focus type. This paper is devoted to finding lower bounds for $M(n)$ for some concrete n by studying the cyclicity of different weak-foci. Since a weak-focus with high order is the most current way to produce high cyclicity, we search for systems with the highest possible weak-focus order. For even n, the studied polynomial system of degree n was the one obtained by [20] where the highest weak-focus order is n^2+n-2 for $n=4,6,\ldots,18$. Moreover, we provide a system which has a weak-focus with order $(n-1)^2$ for $n\leq 12$. We show that Christopher's approach [5], aiming to study the cyclicity of centers, can be applied also to the weak-focus case. We also show by concrete examples that, in some families, this approach is so powerful and the cyclicity can be obtained in a simple computational way. Finally, using this approach, we obtain that $M(6)\geq 39, M(7)\geq 34$ and $M(8)\geq 63$.

1. Introduction and statement of the main results

The second part of the sixteenth Hilbert's problem is concerned with the maximal number (denoted by $H(n)$) and relative positions of the limit cycles of planar polynomial systems of degree n

$$(\dot{x},\dot{y})=(P_n(x,y),Q_n(x,y)). \tag{1}$$

This problem remains unsolved even for quadratic systems. There have been a lot of attempts to make progress in this problem, see the survey articles of Ilyashenko and Li, [12, 14]. Many results have been obtained for lower bounds of $H(n)$. For example, $H(2)\geq 4$, $H(3)\geq 13$, $H(4)\geq 21$, $H(5)\geq 28$, and $H(6)\geq 35$. Recently, Johnson in [13] show numerically that $H(4)\geq 26$. Han and Li detail other lower bounds in [10]. In particular, developing the method used in [6] and introducing some new perturbation techniques, they improve all the existing results for $n\geq 7$ and prove that $H(n)$ grows at least as fast as $(n+2)^2\log(n+2)/\log 4$, see also [10].

We note that the standard technique to obtain a high number of limit cycles for a polynomial differential system is the perturbation of symmetric polynomial systems or systems with many centers or weak-foci. But there are relative few results concerning on the maximum number of limit cycles surrounding only one singularity. In the first paragraph of his paper [27], H. Zoladek said "The particular version of this (Hilbert's 16th) problem is to estimate the number $M(n)$ of small amplitude limit cycles bifurcating from an elementary center or an elementary focus. The ... problem is ... still complicated." Clearly $M(n)\leq H(n)$.

Bautin [3] proved that $M(2)=3$. For cubic systems without quadratic terms, Sibirskii in [24] proved that no more than five limit cycles could be bifurcated from one critical point. Zoladek found an example where 11 limit cycles could be bifurcated from a single critical point of a cubic system, see [27, 28]. Christopher in [5] gave a simpler proof of Zoladek's result perturbing a Darboux cubic center. The same lower bound was also done with a different Darboux cubic center by Bondar and Sadovskii in [4]. By perturbations of a family of Darboux quartic (resp. quintic) systems inside the general quartic (resp.

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