# ALGEBRAIC SOLUTIONS AND INTEGRABILITY FOR PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT

The paper presents a short survey about the Darboux integrability theory of planar polynomial vector fields and states two open problems.

## 1. Introduction

By definition a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P$ and $Q$ are polynomials with coefficients, in $\mathbb{F}$ where $\mathbb{F}$ will denote in the rest of the note either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. In this note we only consider polynomial systems (1) such that $P$ and $Q$ are relatively prime. In other words, we only consider polynomial systems (1) having finitely many singular points.

This work wants to show relationships between the theory of limit cycles and algebraic curves for polynomial differential equations in the real plane. Indeed, already in 1900 such relationships were suggested by the way that Hilbert in [8] stated his 16th problem into two parts: the first one about the topology of real algebraic curves and the second part about the maximum number of limit cycles of polynomial systems having a given degree.

The links between integrability and algebraic solutions are more clear and better known. In 1878 Darboux [6] showed how the first integrals of polynomial systems possessing sufficient algebraic solutions are constructed (see Darboux Theory in Section 2). In particular, he proved that if a polynomial system of degree $m$ has at least $m(m+1)/2$ algebraic solutions, then it has a first integral. Recently, this link appeared in the theory of the center for quadratic systems (see the work of Schlomiuk [14, 15, 16]. But this link is not restricted to quadratic systems and there is a wide literature going back some years on this question, a general reference is the paper of Pearson, Lloyd and Christopher [12].

This paper first presents a short survey about the Darboux theory. As a mather of fact the best two improvements to Darboux Theorem are due to Jouanoulou [9] in 1979 and to Prelle and Singer [13] in 1983. The first showing