INTEGRABILITY AND ALGEBRAIC SOLUTIONS FOR PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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Introduction

By definition a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are polynomials with coefficients, in $\mathbf{F}$ where $\mathbf{F}$ will denote in the rest of the note either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. In this note we only consider polynomial systems (1) such that P and Q are relatively prime. In other words, we only consider polynomial systems (1) having finitely many singular points.

This note contributes to show the link between the theory of polynomial systems and the algebraic curves. Indeed, already in 1878, Darboux [6] showed how the first integrals of polynomial systems possessing sufficient algebraic solutions are constructed. In particular, he proved that if a polynomial system of degree m has at least $m(m+1)/2$ algebraic solutions, then it has a first integral. On the other hand such links were also suggested in 1900 by the way that Hilbert in [10] stated his 16th problem into two parts: the first one about the topology of real algebraic curves and the second part about the maximum number of limit cycles of polynomial systems having a given degree. Recently, this link appeared in the theory of the center for quadratic systems (i.e. $m = 2$), see the work of Schlomiuk [17, 18, 19]. But this link is not restricted to quadratic systems and there is a wide literature going back some years on this question, a general reference is the paper of Pearson, Lloyd and Christopher [15].

This note first presents a short survey about the Darboux method. As a mather of fact the best two improvements to Darboux Theorem are due to Jouanoulou [11] in 1979 and to Prelle and Singer [16] in 1983. The first showing that if the number of algebraic solutions of a polynomial system of degree m is at least $2+[m(m+1)/2]$, then the system has a rational