On the degree of the invariant algebraic curves of polynomial differential systems

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Abstract

We present the two recent examples (of Christopher and Llibre [7] and Moulin Ollagnier [11]) on the existence of invariant algebraic curves of arbitrary degree for quadratic polynomial differential systems without rational first integrals, which solved in the negative sense an open question of Lins Neto [9]. We provide a new proof of the result of Moulin Ollagnier with the explicit expression of its invariant algebraic curves (unknown previously). Finally, we present an open question.

By definition a *planar polynomial differential system* or simply a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

or equivalently

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)},$$

where P and Q are polynomials in the variables x and y. Moreover, the dependent variables x and y, the independent variable t, and the coefficients of the polynomials P and Q are either all complex, or all real. In the former case the system is called a *complex* polynomial system and in the later a *real* polynomial system. The independent variable t will be called the *time* of system (1). The *degree* m of the polynomial system (1) will be the maximum of the degrees of the polynomials P and Q.

Let $f \in \mathbb{C}[x, y]$; i.e. f is a polynomial with complex coefficients in the variables x and y. The complex algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of the vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{C}[x, y]$ we have

$$\mathcal{X} f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree m, then any cofactor has at most degree $m - 1$.

We remark that in the definition of invariant algebraic curve $f = 0$ we always allow that this curve is complex; that is $f \in \mathbb{C}[x, y]$. As we will see this is due to the fact