# A NOTE ON THE DZIOBECK CENTRAL CONFIGURATIONS

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Abstract. For the Newtonian $n$–body problem in $\mathbb{R}^{n-2}$ with $n \geq 3$ we prove that the following two statements are equivalent.

(a) Let $x$ be a Dziobek central configuration having one mass located at the center of mass.

(b) Let $x$ be a central configurations formed by $n-1$ equal masses located at the vertices of a regular $(n-2)$–simplex together with an arbitrary mass located at its barycenter.

## 1. Introduction and statement of the main results

The main problem of the classical Celestial Mechanics is the $n$-*body problem*; i.e. the description of the motion of $n$ particles of positive masses under their mutual Newtonian gravitational forces. This problem is completely solved only when $n = 2$, and for $n > 2$ there are only few partial results.

Consider the Newtonian $n$–body problem in the $d$–dimensional space $\mathbb{R}^d$, i.e.

$$\ddot{x}_i = \sum_{j=1,\ j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3}, \quad \text{for} \quad i = 1, \ldots, n.$$

Here $m_i$ are the masses of the bodies, $x_i \in \mathbb{R}^d$ are their positions, and $r_{ij} = |x_i - x_j|$ are their mutual distances. The vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{nd}$ will be called the *configuration* of the system. The differential equations are well–defined if the configuration is of non–collision type, i.e, $r_{ij} \neq 0$ when $i \neq j$. The dimension of any non–collision configuration of $n \geq 2$ bodies satisfies $1 \leq \delta(x) \leq n-1$.

We define the *dimension* $\delta(x)$ *of a configuration* $x$ to be the dimension of the smallest affine subspace of $\mathbb{R}^d$ which contains all of the points $x_i$. As usual configurations with $\delta(x) = 1, 2, 3$ will be called *collinear, planar* and *spatial*, respectively.

The *total mass* and the *center of mass* of the $n$ bodies are

$$M = m_1 + \ldots + m_n, \qquad c = \frac{1}{M}\left(m_1 x_1 + \cdots + m_n x_n\right),$$

respectively. A configuration $x$ is a *central configuration* if the acceleration vectors of the bodies satisfy

$$\sum_{j=1,\ j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3} + \lambda(x_i - c) = 0, \quad \text{for} \quad i = 1, \ldots, n, \tag{1}$$

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Central configurations started to be studied in the second part of the 18th century, there is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in the books of Wintner [17] and Hagihara [6]. For a modern background see, for instance, the papers of Albouy and Chenciner [2], Albouy and Kaloshin [3], Hampton and Moeckel [7], Moeckel [9], Palmore [13], Saari [14], Schmidt [15], Xia [18], ... One of the reasons why central configurations are important is that they allow to obtain the unique explicit solutions in function of the time of the $n$–body problem known until now, the *homographic solutions* for which the ratios of the mutual distances between the bodies remain constant. They are also important because the total collision or the total parabolic escape at infinity in the $n$–body problem is asymptotic to central configurations, see for more details Dziobek [5] and [14]. Also if we fix the total energy $h$ and the angular momentum $c$ of the $n$–body problem, then some of the bifurcation points $(h, c)$ for the topology of the level sets with energy $h$ and angular momentum $c$ are related with the central configurations, see Meyer [11] and Smale [16] for a full background on these topics.

Moulton [12] proved that for a fixed mass vector $m = (m_1, \dots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration, up to translation and scaling.

At the other extreme of the dimension range, Lagrange [8] showed that for $n = 3$, the only central configuration $x$ with $\delta(x) = n-1 = 2$ is the equilateral triangle, and it is central for all choices of the masses. An analogous result of Lagrange's result holds for all $n$. Thus, it is well known that for $n \geq 3$ the only central configuration $x$ with $\delta(x) = n - 1$ of the $n$–body problem is formed by the vertices of a regular $(n - 1)$–simplex, which is central for all choices of the masses. Of course, a 1–simplex is a closed interval, a 2–simplex is a closed equilateral triangle, a 3–simplex is a closed regular tetrahedron, and so on.

For other values of the dimension $\delta(x)$ the problem of finding or even counting the central configurations $x$ of the $n$–body problem is very difficult, see for instance Moeckel [10]. The dimension $d(x) = 2$ is of course the most interesting of all, because planar central configurations give rise to physically realistic periodic orbits. For $n = 4$, Dziobek [5] formulated the planar central configuration problem in terms of mutual distances $r_{ij}$ and obtained algebraic equations characterizing the central configurations. His approach has been adopted and developed by Albouy [1] in his study of the central configurations with four equal masses, see also Albouy and Llibre [4]. The natural generalization to higher $n$ is the case $\delta(x) = n-2$. Following [1] and [10] we call such central configurations *Dziobek configurations.*

The goal of this paper is to prove the following result.

**Theorem 1.** *The following two statements are equivalent for the $n$–body problem with $n \geq 3$.*

(a) *Let $x$ be a central configuration with $\delta(x) = n - 2$ (i.e. a Dziobek central configurations) having one mass located at the center of mass.*

(b) *Let $x$ be a central configurations formed by $n-1$ equal masses located at the vertices of a regular $(n-2)$–simplex together with an arbitrary mass located at its barycenter.*

Of course, statement (b) of Theorem 1 implies immediately statement (a). The converse implication is proved in section 3.

If the central configuration is not Dziobek, then the equivalence of Theorem 1 does not hold. Thus, for instance, consider a regular polygon with $n-1>3$ equal masses located at its vertices and an arbitrary mass located at its barycenter. This is a non–Dziobek central configuration having one mass located at the center of mass, and since $n>4$ it is different from the configuration formed by the vertices of a regular $(n-2)$–simplex together with its barycenter.

## 2. Equations for the Dziobek central configurations

Since we want to study the Dziobek central configurations we consider the $n$–body problem in $\mathbb{R}^{n-2}$. To each configuration $x=(x_1,\ldots,x_n)\in\mathbb{R}^{n(n-2)}$ we associate the $n\times n$ matrix

$$X=\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let $X_k$ be the $(n-1)\times(n-1)$ matrix obtained delating from the matrix $X$ its $k$–th column and its last row. Then define $\Delta_k=(-1)^{k+1}\det(X_k)$ for $k=1,\ldots,n$.

Dziobek [5] (see also equations (8) and (16) of Moeckel [10]) reduces the equations for the central configurations (1) of the $n$–body problem to the following system of

$$N=\frac{n(n-1)}{2}+n+2$$

equations and $N$ unknowns:

$$\begin{aligned} \frac{1}{r_{ij}^3} &= c_1+c_2\frac{\Delta_i\Delta_j}{m_im_j}, \\ t_i-t_j &= 0, \end{aligned} \tag{2}$$

for $1\le i<j\le n$, with

$$t_i=\sum_{j=1,\,j\neq i}^{n}\Delta_j\,r_{ij}^2.$$

The $N$ unknowns in equations (2) are the $n(n-1)/2$ mutual distances $r_{ij}$, the $n$ variables $\Delta_i$, and the two constants $c_k$.

## 3. Proof of Theorem 1

In order to complete the proof of Theorem 1, we must prove that statement (a) of that theorem implies statement (b).

We assume that $x=(x_1,\ldots,x_n)\in\mathbb{R}^{n(n-2)}$ is a Dziobek central configuration having one mass located at the center of masses. Without loss of generality we can suppose:

(i) the center of mass is at the origin of coordinates;
(ii) the mass $m_n$ is located at the center of mass, i.e. $x_n=0$;
(ii) the unit of mass it taken in such a way that $m_{n-1}=1$. Then

$$x_{n-1}=-\sum_{i=1}^{n-2}m_ix_i.$$

Since $x$ is a Dziobek central configuration it satisfies the equations (2). Easy computations using the properties of the determinants show that

$$
\begin{aligned}
&\frac{\Delta_i \Delta_j}{m_i m_j} = \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad 1 \le i < j \le n-1, \\
(3) \qquad &\frac{\Delta_i \Delta_n}{m_i m_n} = \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad i = 1, \dots, n-1.
\end{aligned}
$$

Then, the first equations of (2) become

$$
\begin{aligned}
&\frac{1}{r_{ij}^3} = c_1 + c_2 \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad 1 \le i < j \le n-1, \\
(4) \qquad &\frac{1}{r_{in}^3} = c_1 + c_2 \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad i = 1, \dots, n-1.
\end{aligned}
$$

The equations $t_i = t_j$ of (2) are trivially satisfied by direct computations, but they are not relevant in this proof.

From equations (4) we obtain that

$$
\begin{aligned}
&r_{ij} = k_1 \quad \text{for} \quad 1 \le i < j \le n-1, \\
(5) \qquad &r_{in} = k_2 \quad \text{for} \quad i = 1, \dots, n-1,
\end{aligned}
$$

where $k_1$ and $k_2$ are constants. Therefore, from the first equations of (5) it follows that the masses $m_k$ for $k = 1, \dots, n-1$ are at the vertices of a regular $(n-2)$–simplex, and from the second equations of (5) we obtain that the mass $m_n$ is at the barycenter of this regular $(n-2)$–simplex. Moreover, since the barycenter must be the center of mass of the $n-1$ masses located at the vertices of a regular $(n-2)$–simplex, this forces that these $n-1$ masses must be equal. Of course, the mass $m_n$ located at the barycenter is arbitrary. This completes the proof of the theorem.

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