Topology and its Applications 159 (2012) 2228–2231

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Periodic point free continuous self-maps on graphs and surfaces

Jaume Llibre

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Catalonia, Spain

ARTICLE INFO

Article history:
Received 5 February 2012
Received in revised form 29 February 2012
Accepted 6 March 2012

MSC:
37C05
37C25
37C30

Keywords:
Periodic point free
Lefschetz zeta function
Lefschetz number
Connected compact graph
Connected compact surface
Orientable surface
Non-orientable surface

ABSTRACT

We prove the following three results. We denote by $\mathrm{Per}(f)$ the set of all periods of a self-map f.

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G},\mathbb{Q}) = r$, and let $f:\mathbb{G}\to\mathbb{G}$ be a continuous map. If $\mathrm{Per}(f)=\emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r-1$, where f_{*1} is the induced action of f on the first homological space.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geqslant 0$ with $b \geqslant 0$ boundary components, and let $f:\mathbb{M}_{g,b}\to\mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b=0$. If $\mathrm{Per}(f)=\emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g-2$ if $b=0$; and 1 and 0, this last with multiplicity $2g+b-2$ if $b>0$.

Let $\mathbb{N}_{g,b}$ be a non-orientable connected compact surface of genus $g \geqslant 1$ with $b \geqslant 0$ boundary components, and let $f:\mathbb{N}_{g,b}\to\mathbb{N}_{g,b}$ be a continuous map. If $\mathrm{Per}(f)=\emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $g+b-2$.

The tools used for proving these results can be applied for studying the periodic point free continuous self-maps of many other compact absolute neighborhood retract spaces.

1. Introduction and statement of the main results

A *discrete dynamical system* $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f:\mathbb{M}\to\mathbb{M}$. A point x is called *fixed* if $f(x)=x$, and *periodic* of *period* k if $f^k(x)=x$ and $f^i(x)\neq x$ if $0<i<k$. We denote the *set of periods* of all the periodic points of f by $\mathrm{Per}(f)$.

The set $\{x, f(x), f^2(x), \dots, f^n(x), \dots\}$ is called the *orbit* of the point $x\in\mathbb{M}$, where f^n means the composition of f with itself n times. To study *the dynamics of a map* f is to study all the different kind of orbits of f. If x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \dots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

In general the periodic orbits play a main role in the dynamics of a discrete dynamical system, for studying them we can use topological information. Probably the best known results in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self-maps on the interval, see [12].

All the non-completely standard notions which will appear in the rest of the introduction will be defined in Section 2.

In this note $\mathbb{M}$ will be either a connected compact graph, or a connected compact surface with or without boundary, orientable or not. Our aim is to study the continuous maps $f:\mathbb{M}\to\mathbb{M}$ having their $\mathrm{Per}(f)=\emptyset$, i.e. *periodic point free* continuous self-maps of $\mathbb{M}$.

Since the word "graph" has several different meanings in mathematics, we shall define here our graphs. A *graph* is a union of *vertices* (points) and *edges*, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertexes (not necessarily different) and the interiors of the edges are disjoint from the vertices.

E-mail address: jllibre@mat.uab.cat.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2012.03.005