ON THE 16–HILBERT PROBLEM

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ABSTRACT. We present a brief survey of recent results on the second part of the 16–th Hilbert problem. We put special emphasis in the algebraic limit cycles.

1. INTRODUCTION

In this brief survey we only consider differential equations in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are polynomials of degree at most d. We recall that a *limit cycle* of the differential equation (1) is a periodic orbit of this equation isolated in the set of all periodic orbits of equation (1).

The notion of limit cycle appears in the years 1891 and 1897 in the works of Poincaré [33]. Moreover, he proved that a polynomial differential equation (1) without saddle connections has finitely many limit cycles, see [33].

Hilbert [16] at the Second International Congress of Mathematicians, celebrated in Paris in 1900, proposed a list of 23 relevant problems for being solved during the XX century. The 16–th problem of the list is:

16. *Problem of the topology of algebraic curves and surfaces*

The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6th order, I have satisfied myself–by a complicated process, it is true–that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches

2010 *Mathematics Subject Classification.* 34Cxx.

Key words and phrases. limit cycles, configuration of limit cycles, polynomial vector fields, Liénard polynomial differential equations, algebraic limit cycles, inverse integrating factor.

when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4th order in three dimensional space can really have.

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the nth degree in x and y. Written homogeneously, this is

$$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0,$$

where X, Y, and Z are rational integral homogeneous functions of the nth degree in x, y, z, and the latter are to be determined as functions of the parameter t.

It is clear that the 16–th Hilbert problem is formulated in two parts. The first part is about the mutual disposition of the maximal number of separate branches of an algebraic curve, and its extension to non-singular real algebraic varieties. In the second part he asked for the maximal number and relative position of the limit cycles of the differential system (1). Usually the first part of the 16–th Hilbert problem is studied by researchers in real algebraic geometry, while the second part is considered by mathematicians working in dynamical systems or differential equations. Hilbert also pointed out that there exist possible connections between these two parts. Some of these connections are described in the survey about the 16–th Hilbert problem written by Jibin Li, see [23].

In what follows when we talk about the 16–th Hilbert problem we always are talking on the second part of the 16–th Hilbert problem.

In 1988 Noel Lloyd [30] observed with respect the 16–th Hilbert problem that *the striking aspect is that the hypothesis is algebraic, while the conclusion is topological.*

Arnold in 1977 and 1983 (see [1] and [2], respectively) stated the *weakened, infinitesimal or tangential* 16*–th Hilbert problem* which we do not consider here, but there are excellent surveys for this modified problem, see for instance the survey of Ilyashenko [19] on the 16–th Hilbert problem, the already mentioned survey of Jibin Li, or the book of Colin Christopher and Chengzhi Li [7], or the survey due to Kaloshin [21], or the one of Yakovenko [40], or more recently the work of Binyamini, Novikov and Yakovenko [4], ...

According with Smale [36] except for the Riemann hypothesis, the second part of the 16–th Hilbert problem seems to be the most elusive of the Hilbert's problems. Smale states the following modern version of the second half of 16–th Hilbert problem:

Consider the polynomial differential equation (1) *in* $\mathbb{R}^2$*. Is there a bound* K *on the number of limit cycles of the form* $K \leq d^q$ *where* d *is the maximum of the degrees of* P *and* Q*, and* q *is a universal constant*?

The possible distribution or topological configurations of limit cycles mentioned as position for Hilbert has also interested to many authors. Coleman in his work [9] on the 16–Hilbert problem said: *For* $d > 2$ *the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes, or eyes enclosing more than a single critical point, can exist.* Here "eye" means a nest of limit cycles. We shall see later on that some of the questions on the possible topological configurations of limit cycles realized by polynomial differential equations can be solved easily.

Another problem very related with the 16–th Hilbert problem is the study of the possible bifurcations of limit cycles. Again this problem will not be considered here, see good information about it in the survey of Jibin Li, or the books of Christopher and Chengzhi Li, Yankian Ye [41], Zhifen Zang et al. [43], ...

Our approach to the 16–Hilbert problem is done through the following seven problems:

Problem 1: *Is it true that a polynomial differential equation* (1) *has a finite number of limit cycles* ?

Problem 2: *Is it true that the number of limit cycles of a polynomial differential equation* (1) *is bounded by a constant depending only on the degree of the polynomials* ?

If the problem 2 has a positive answer then its bound is denoted by $H(d)$, and called the *Hilbert number* for the polynomial differential equations (1) of degree d.

Problem 3: *If the problem 2 has a positive answer, provide an upper bound for $H(d)$.*

Smale [36] in 1998 said that the 16–Hilbert problem looks very difficult, and that first we must consider a special class of simpler polynomial differential equations, and he propose to study the 16–Hilbert problem restricted to the *Liénard polynomial differential equations*, i.e. to the polynomial differential equations of the form

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{2}$$

where $F(x)$ is a polynomial in the variable x of degree d.

Problem 4: *What about the problems* 2 *and* 3 *if we restrict the study to the Liénard polynomial differential equations* (2) ?

For the Liénard polynomial differential equations we do not talk about the problem 1 because as we shall see the problem 1 has been solved in positively for all polynomial differential equations (1).

Problem 5: *What are the possible topological configurations of limit cycles for the polynomial differential equations* (1) *?*

An *algebraic limit cycle* is an oval of an algebraic curve which is a limit cycle of a polynomial differential equation (1).

Problem 6: *Is it true that the number of algebraic limit cycles of a polynomial differential equation* (1) *is bounded by a constant depending only on the degree of the polynomials* ?

If the problem 6 has a positive answer then its bound is denoted by $H^a(d)$, and we called it the *algebraic Hilbert number* for the polynomial differential equation (1) of degree d.

Problem 7: *If the problem 6 has a positive answer, provide an upper bound for $H^a(d)$.*

The first four problems have considered by several authors, see for instance the surveys of Ilyashenko and of Jibin Li. Here, we pass fast for these first four problems, and we shall dedicate more space to the last three problems which as far as we know there has not been considered for the moment in any other survey.

2. Problem 1

Dulac [12] in 1923 claimed that any polynomial differential equation (1) always has finitely many limit cycles. Ilyashenko [17] in 1985 found an error in Dulac's paper. Later on, two long works have appeared, independently, providing proofs of Dulac's assertion, one due to Écalle

[14] in 1992 and the other to Ilyashenko [18] in 1991. As Smale mentioned in [36] these two papers have yet to be thoroughly digested by the mathematical community.

Bamon [3] in 1986 proved that any polynomial differential equation of degree 2 has finitely many limit cycles. His result uses previous results of Ilyashenko.

From the work of Dulac [12] it follows that and if a polynomial differential equation (1) has saddle connections forming a simple homoclinic or heteroclinic loop, then also the equation has finitely many limit cycles, see for more details the nice work of Sotomayor [37]. Here a *homoclinic* or *heteroclinic loop* is formed by $k = 1$ or $k > 1$ saddles (eventually some saddles can be repeated) and k different separatrices connecting these saddles and forming a loop (eventually some points of this loop can be identified in a repeated saddle) in such a way that at least in one of the two sides of the loop is defined a Poincaré return map. Let $\mu_i < 0 < \lambda_i$ the eigenvalues of these saddles, if

$$\prod_{i=1}^{k} \frac{\lambda_i}{\mu_i} \neq 1,$$

then the loop is called *simple*.

3. Problem 2

Since the polynomial differential equations of degree 1 or linear differential equations have no limit cycles, it follows that the Hilbert number $H(1) = 0$.

Unfortunately we do not know if an uniform upper bound for the maximum number of limit cycles exists for all polynomial vector fields of degree d if $d \geq 2$.

4. Problem 3

Since problem 2 remains open for degree $d \geq 2$, we do not know if the Hilbert number $H(d)$ exist for $d \geq 2$.

In 1957 Petrovskii and Landis [31] claimed that the polynomial differential equations of degree $d = 2$ has at most 3 limit cycles, i.e. that the Hilbert number $H(2) = 3$. Soon (in 1959) a gap was found in the arguments of Petrovskii and Landis see [32]. Later on Lan Sun Chen and Ming Shu Wang [5] in 1979 and Songling Shi [35] in 1982 provided the first polynomial differential equations of degree 2 having 4 limit cycles, and consequently showing that $H(2) \geq 4$.

Some lower bounds for $H(d)$ have been given, mainly by Christopher and Lloyd [8] and Jibin Li, see the survey of this last author who analyze these lower bounds.

5. Problem 4

The study of Liénard differential equations (not necessarily polynomial) has a long history and a lot of results were obtained on them, see for example the book [43].

If $F(x) = x^3 - x$ then the Liénard differential equation (2) is the famous van der Pol's equation which has at most one limit cycle.

Van der Pol in 1926, Liénard in 1928 and Andronov in 1929 proved that the periodic solution of a self–sustained oscillation in a vacuum tube was a limit cycle in the sense defined by Poincaré. After this observation of the existence of a limit cycle in the nature, the existence, non–existence, uniqueness and other properties of the limit cycles have been intensively studied not only by the mathematicians, which were already motivated by the works of Poincaré and Hilbert, also by the physiciens, and later on by the chemists, biologists, economists, and many others. The limit cycles started to be important in the sciences.

For the Liénard polynomial differential equations (2) of degree d the existence of a uniform bound for the maximum number of limit cycles also remains unproved. But when the degree n of these systems is odd Ilyashenko and Panov in [20] obtained an uniform upper bound for the number of limit cycles in a subclass of systems such that the polynomial $F(x)$ is monic and its coefficients satisfy some estimations.

In 1977 Lins, de Melo and Pugh conjectured in [24] that the Liénard polynomial differential equation (2) of degree $d \geq 3$ has at most $[(d-1)/2]$ limit cycles, where $[(d-1)/2]$ means the largest integer less than or equal to $(d-1)/2$. Moreover, they provide Liénard polynomial differential equations (2) for any degree $d \geq 3$ having at least $[(d-1)/2]$ limit cycles. They also proved that the conjecture is true for $d = 3$. It is not difficult to show that their conjecture also holds for the degrees $d = 1, 2$.

In 2007 Dumortier, Panazzolo and Roussarie [13] gave a counterexample to this conjecture for $d = 7$ and mentioned that it can be extended to $d \geq 7$ odd. Recently, de Maesschalck and Dumortier proved in [11] that the Liénard polynomial differential equation (2) of degree $d \geq 6$ can have $[(d-1)/2] + 2$ limit cycles. In the last two papers the results are proved using singular perturbation theory, and the authors work with relaxation oscillation solutions to study the number of limit cycles.

Chengzhi Li and Llibre [22] shows in 2012 that the Lins–de Melo–Pugh's conjecture is true for the Liénard polynomial differential equations of degree $d = 4$. So at this moment only remains open the conjecture for degree $d = 5$.

6. Problem 5

A *topological configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Given a topological configuration of limit cycles $C = \{C_1, \ldots, C_n\}$ the curve C_i is *primary* if there is no curve C_j of C contained into the bounded region limited by C_i.

Two topological configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are (*topologically*) *equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^{n} C_i\right) = \left(\cup_{i=1}^{m} C'_i\right)$. Of course, for equivalent configurations of limit cycles C and C' we have that $n = m$.

We say that a polynomial differential equation (1) *realizes* the configuration of limit cycles C if the set of all limit cycles of X is equivalent to C.

In 2004 Llibre and Rodríguez [28] proved the following result.

Theorem 1. *Let $C = \{C_1, \ldots, C_n\}$ be a topological configuration of limit cycles, and let r be its number of primary curves. Then the following statements hold.*

(a) *The configuration C is realizable by some polynomial differential equation.*
(b) *The configuration C is realizable as algebraic limit cycles by a polynomial differential equation of degree $\leq 2(n + r) - 1$. Moreover, such a polynomial differential equation has a first integral of Darboux type.*

Of course, statement (a) of Theorem 1 follows immediately from statement (b).

Statement (a) of Theorem 1 was solved by first time by Schecter and Singer [34] and Sverdlove [38], but they do not provide an explicit polynomial vector field satisfying the given configuration of limit cycles, as it was provided in the proof of statement (b) of Theorem 1.

Christopher [6] in 2001 proved the following result. If $f = f(x, y)$ is a polynomial we denote its partial derivatives with respect to the variables x and y as f_x and f_y, respectively.

Theorem 2. *Let $f = 0$ be a non–singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the straight line $D = 0$ lies outside all bounded components of $f = 0$. Choose the constants α and β so that $\alpha D_x + \beta D_y \neq 0$, then the polynomial differential equation of degree m,*

$$\dot{x} = \alpha f - D f_y, \qquad \dot{y} = \beta f + D f_x,$$

has all the bounded components of $f = 0$ as hyperbolic limit cycles. Furthermore, the differential equation has no other limit cycles.

Theorem 2 improves a similar result due to Winkel [39], but the polynomial differential equation obtained by Winkel has degree $2m-1$.

Given a topological configuration of n limit cycles we can consider an equivalent topological configuration formed by circles. Then, consider the algebraic curve $f = 0$ formed by the product of all the circles. Applying Theorem 2 to the curve $f = 0$, we obtain a polynomial differential equation of degree $2n$ which realizes the given topological configuration of n limit cycles with algebraic limit cycles. A difference between the polynomial differential equations of Theorems 1 and 2, is that the first always has a first integral, and the second, in general, has no first integrals.

In short, both theorems show that any topological configuration of limit cycles is realizable with algebraic limit cycles for some polynomial differential equation, and provide the degree of such polynomial differential equations. But there are many questions which remains open, as for instance: *what are the possible topological configurations realizable for the polynomial differential equations of a given degree*? Of course this question is strongly more difficult than the question to provide a uniform upper bound for the maximum number of limit cycles that the polynomial differential equations of a given degree can have.

7. Problems 6 and 7

Associated to the polynomial differential equation (1) there is the *polynomial vector field*

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{3}$$

The algebraic curve $f(x,y)=0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the polynomial vector field $\mathcal{X}$ or of the polynomial differential equation (1) if for some polynomial $K \in \mathbb{C}[x,y]$ we have

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{4}$$

Since on the points of an algebraic curve $f=0$ the gradient $(\partial f/\partial x, \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X}$ (see (4)), the vector field $\mathcal{X}$ is tangent to the curve $f=0$. Hence the curve $f=0$ is formed by orbits of the vector field $\mathcal{X}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f=0$ satisfying (4) for some polynomial K, because it is *invariant* under the flow defined by $\mathcal{X}$.

The next well known result tell us that we can restrict our attention to the irreducible invariant algebraic curves, for a proof see for instance [25]. Here, as it is usual, $\mathbb{R}[x,y]$ denotes the ring of all polynomials in the variables x and y and coefficients in $\mathbb{R}$.

Proposition 3. *We suppose that $f \in \mathbb{R}[x,y]$ and let $f = f_1^{n_1}\cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{R}[x,y]$. Then for a polynomial vector field $\mathcal{X}$, $f=0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i=0$ is an invariant algebraic curve for each $i=1,\dots,r$ with cofactor K_{f_i}. Moreover $K_f = n_1K_{f_1}+\ldots+n_rK_{f_r}$.*

Consider the space Σ' of all real polynomial vector fields (4) of degree d having real irreducible invariant algebraic curves. A simpler version of the second part of the 16th Hilbert's problem is: *Is there an upper bound on the number of algebraic limit cycles of any polynomial vector field of Σ'?* Now we cannot provide an answer to this question for general real algebraic curves, but we give the answer for the following class of algebraic curves.

We say that a set $f_j=0$, for $j=1,\dots,k$, of irreducible algebraic curves is *generic* if it satisfies the following five conditions:

(i) There are no points at which $f_j=0$ and its first derivatives all vanish (i.e. $f_j=0$ is a non–singular algebraic curve).
(ii) The highest order homogeneous terms of f_j have no repeated factors.
(iii) If two curves intersect at a point in the affine plane, they are transversal at this point.
(iv) There are no more than two curves $f_j=0$ meeting at any point in the affine plane.
(v) There are no two curves having a common factor in the highest order homogeneous terms.

The next result was proved by Llibre, Ramírez and Sadovskaia [26] in 2010.

Theorem 4. *For a polynomial vector field $\mathcal{X}$ of degree $d \geq 2$ having all its irreducible invariant algebraic curves generic, the maximum number of algebraic limit cycles is at most $1+(d-1)(d-2)/2$ if d is even, and $(d-1)(d-2)/2$ if d is odd. Moreover these upper bounds are reached.*

For cubic polynomial vector fields having all their irreducible invariant algebraic curves generic Theorem 4 says that one is the maximum number of algebraic limit cycles, but there are examples of cubic polynomial vector fields having two algebraic limit cycles, of course such vector fields have non–generic invariant algebraic curves. Thus the system

$$\dot{x} = 2y(10 + xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3,$$

has two algebraic limit cycles contained into the invariant algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$, see Proposition 19 of [29].

Up to now all the polynomial vector fields having non–generic invariant algebraic curves and more algebraic limit cycles than the upper bounds given in Theorem 4 for the generic case have degree odd, and at most one limit cycle than the upper bound of Theorem 4. So, in [26] we did the following conjecture.

Conjecture 1. *The algebraic Hilbert number is*

$$H^a(d) = 1 + (d-1)(d-2)/2.$$

The easiest version of this conjecture is it restriction to the polynomial vector fields of degree 2.

Conjecture 2. $H^a(2) = 1$.

Note that both conjectures are true when d is even and we restrict the algebraic limit cycles to generic invariant algebraic curves.

An interesting result on the limit cycles of a C^1 differential equation in the plane is the following one due to Giacomini, Llibre and Viano [15], see an easier proof in [28]. This result has been used in the proofs of Theorems 1 and 4. First we need a definition.

Let U be an open subset of $\mathbb{R}^2$. A function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of a C^1 vector field $\mathcal{X}$ defined on U if V verifies the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V$$

in U.

Theorem 5. *Let X be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V : U \to \mathbb{R}$ be an inverse integrating factor of X. If γ is a limit cycle of X, then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

Acknowledgements

The author is partially supported by a MICINN/FEDER grant number MTM2008–03437, by an AGAUR grant number 2009SGR–410 and by ICREA Academia.

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