

# BRIEF SURVEY ON THE TOPOLOGICAL ENTROPY

JAUME LLIBRE

ABSTRACT. In this paper we give a brief view on the topological entropy. The results here presented are well known to the people working in the area, so this survey is mainly for non-experts in the field.

## CONTENTS

1. Introduction	1
2. The topological entropy	2
3. Part I. Topological entropy in one-dimensional spaces	3
3.1. Entropy of piecewise monotone interval maps	3
3.2. Entropy and horseshoes for interval maps	3
3.3. Continuity properties of the entropy	4
3.4. Semiconjugacy to constant slope maps	5
3.5. Entropy for circle maps	5
3.6. Entropy for graph maps	6
4. Part II. Topological entropy in spaces of dimension $> 1$	9
4.1. Entropy and volume growth	9
4.2. Entropy and periodic points	10
4.3. Entropy conjecture	11
4.4. Volume growth and the spectral radius	13
Acknowledgments	13
References	13

## 1. INTRODUCTION

We do not try to be exhaustive on all the result about the topological entropy, thus here we do not consider or do not put too much attention on its relation with the metric entropy, the local entropy, Lyapunov exponents, etc, and we do not say anything about flows or other actions, nor about generic situations. Also in the case of surfaces there are more results available, because one can use Nielsen–Thurston theory for the study of the global dynamics of homeomorphism, see for example [17], [18], but we want to keep our survey short and relatively easy to

read, and covering all these other aspects we shall need another survey. The results will be presented without proofs, but providing explicit references about them.

The paper has two parts well separated.

The first is dedicated to the topological entropy in one-dimensional spaces, more precisely on the interval, the circle and a graph. For this part the main reference for completing the results here mentioned, additionally to the original papers where they are proved by first time, is the book [4].

In the second part we consider the topological entropy in spaces of dimension larger than one, and for going further in this part see the survey [34] where you can find examples and open questions and it is more detailed the relation of the topological entropy with other dynamical invariants. For more information about the topological entropy see the good surveys [19, 24, 31, 32, 34, 47, 53, 57], or for the holomorphic case [22].

## 2. THE TOPOLOGICAL ENTROPY

Let  $X$  be a metric compact Hausdorff topological space, and let  $f : X \rightarrow X$  a continuous map. By iterating this map we obtain a dynamical system. How to measure its complicated dynamics? How many very different orbits it has? How fast it “mixes” together various sets, etc. This can be measured by the topological entropy.

There are several definitions of topological entropy. The classical definition is due to Adler, Konheim and McAndrew [1]. Here we shall use the definition of Bowen [14] because it is shorter to introduce. For equivalent definitions and properties of the topological entropy see for instance the book of Hasselblatt and Katok [26].

We will also consider the *topological entropy of  $f$* , defined as follows. First define the metric  $d_n$  on  $M$  by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(f^i(x), f^i(y)), \quad \forall x, y \in M.$$

A finite set  $S$  is called  $(n, \varepsilon)$ -separated with respect to  $f$  if for any different  $x, y \in S$  we have  $d_n(x, y) > \varepsilon$ . The maximal cardinality of an  $(n, \varepsilon)$ -separated set is denoted  $S_n$ . Define

$$h(f, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log S_n.$$

Then the topological entropy of  $f$  is

$$h(f) = \lim_{\varepsilon \rightarrow 0} h(f, \varepsilon).$$

Some basic properties of the topological entropy are given in the next two lemmas.

**Lemma 1.** *We have*

$$h(f^n) = n \cdot h(f).$$

**Lemma 2.** *Let  $X$  and  $Y$  be compact Hausdorff spaces,  $f : X \rightarrow X$ ,  $g : Y \rightarrow Y$  and  $\varphi : X \rightarrow Y$  be continuous maps such that  $\varphi \circ f = g \circ \varphi$ .*

- (a) *If  $\varphi$  is injective then  $h(f) \leq h(g)$ .*
- (b) *If  $\varphi$  is surjective then  $h(f) \geq h(g)$ .*
- (c) *If  $\varphi$  is bijective then  $h(f) = h(g)$ .*

Lemmas 1 and 2 are well known; for a proof see for instance [4].

### 3. PART I. TOPOLOGICAL ENTROPY IN ONE-DIMENSIONAL SPACES

**3.1. Entropy of piecewise monotone interval maps.** Let  $I$  be a closed interval. A continuous map  $f : I \rightarrow I$  will be called an *interval map*.

We say that an interval map  $f : I \rightarrow I$  is *piecewise (strictly) monotone* if there exists a finite partition of  $I$  into intervals such that on each element of this partition  $f$  is (strictly) monotone.

**Theorem 3.** *Assume  $f$  is piecewise strictly monotone. Let  $c_n$  be the number of pieces of monotonicity of  $f^n$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log c_n = h(f),$$

*and  $(1/n) \log c_n \geq h(f)$  for any  $n$ .*

Theorem 3 was proved independently by Rothschild [49], Misiurewicz and Szlenk [44] and Young [58].

**3.2. Entropy and horseshoes for interval maps.** If  $f : I \rightarrow I$  is an interval map and  $s \geq 2$ , then an *s-horseshoe* for  $f$  is an interval  $J \subset I$  and a partition of  $J$  into  $s$  closed subintervals  $J_k$  such that  $J \subset f(J_k)$  for  $k = 1, \dots, s$ .

**Proposition 4.** *If  $f$  has an s-horseshoe then*

$$h(f) \geq \log s.$$

Proposition 4 follows from the computations of Adler and McAndrew in [2].

**Theorem 5.** *Assume that the interval map  $f$  has positive entropy. Then there exist sequences  $(k_n)_{n=1}^\infty$  and  $(s_n)_{n=1}^\infty$  of positive integers such that  $\lim_{n \rightarrow \infty} k_n = \infty$ , for each  $n$  the map  $f^{k_n}$  has an  $s_n$ -horseshoe and*

$$\lim_{n \rightarrow \infty} \frac{1}{k_n} \log s_n = h(f).$$

We shall say that  $f$  has a constant slope  $s$  if on each of its pieces of monotonicity it is affine with the slope coefficient of absolute value  $s$ .

**Corollary 6.** *Assume that  $f$  is piecewise strictly monotone and has a constant slope  $s$ . Then  $h(f) = \max(0, \log s)$ .*

Another corollary to Theorem 5 is the following.

**Theorem 7.** *If  $f : I \rightarrow I$  is an interval map, then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Card}\{x \in I : f^n(x) = x\} \geq h(f).$$

Theorems 5 and 7 were proved by Misiurewicz and Szlenk [44] for piecewise monotone maps, and by Misiurewicz [39], [40] in the general case.

Corollary 6 was proved independently by Misiurewicz and Szlenk [44], Young [56] and Milnor and Thurston [38].

**3.3. Continuity properties of the entropy.** A real valued function  $\varphi$  is called *lower* (respectively *upper*) *semi-continuous* if for each point  $x$  we have

$$\liminf_{y \rightarrow x} \varphi(y) \geq \varphi(x) \quad \text{respectively} \quad \limsup_{y \rightarrow x} \varphi(y) \leq \varphi(x).$$

Of course a function is continuous if and only if it is both lower and upper semi-continuous.

**Theorem 8.** *The function  $h(\cdot)$  is lower semi-continuous.*

Theorem 8 is due to Misiurewicz and Szlenk [44] for the piecewise monotone case and due to Misiurewicz [39] for the general case.

**Corollary 9.** *On the space of all  $C^1$  piecewise strictly monotone maps with a given number of pieces of monotonicity, with the  $C^1$  topology, the topological entropy is continuous.*

Corollary 9 is due to Milnor and Thurston [38].

**Corollary 10.** *On the space of all  $C^r$  piecewise monotone maps (where  $r \geq 2$ ) with the  $C^r$  topology, the topological entropy is continuous at all  $f$  for which the critical points are non-degenerate (i.e. there are no points at which both  $f'$  and  $f''$  vanish simultaneously) and no critical point is an endpoint of  $I$ .*

Corollary 10 is due to Bowen [15].

**Theorem 11.** *The topological entropy, as a function from the space of all unimodal maps with the topology of uniform convergence is continuous at all points at which it is positive.*

Theorem 11 is due to Misiurewicz [42].

### 3.4. Semiconjugacy to constant slope maps.

**Theorem 12.** *If  $f$  is piecewise strictly monotone and  $h(f) = \log \beta > 0$  then  $f$  is semiconjugate to some map  $g : [0, 1] \rightarrow [0, 1]$  with constant slope  $\beta$  via a non-decreasing map.*

Theorem 12 was proved by Milnor and Thurston [38] for piecewise strictly monotone maps, see a different proof in [4].

Let  $X$  be a metric space and let  $f : X \rightarrow X$  a continuous map. The map  $f$  is called *(topologically) transitive* if for every pair of open sets  $U$  and  $V$  in  $X$ , there is a positive integer  $n$  such that  $f^n(U) \cap V \neq \emptyset$ .

**Corollary 13.** *If an interval map  $f$  is transitive, then  $h(f) > 0$ .*

Corollary 13 was stated in Blokh [9] without a proof, for a proof see Block and Coven [7].

**Corollary 14.** *If  $f$  is piecewise strictly monotone and transitive, then  $f$  is conjugate to some map  $g : [0, 1] \rightarrow [0, 1]$  with constant slope  $\beta$ , where  $\beta = \exp(h(f))$ .*

Corollary 14 is due to Parry [48].

**3.5. Entropy for circle maps.** Let  $\mathbb{S}^1$  be the circle and  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be a continuous map of degree one, i.e. a *circle map*. Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  denote a lifting of  $f$ , i.e. a map such that  $f \circ e = e \circ F$ , where  $e$  denotes the natural projection from  $\mathbb{R}$  to  $\mathbb{S}^1$  given by  $e(X) = \exp(2\pi i X)$ . We note that  $F$  is not determined uniquely, that is if  $F$  and  $F'$  are two liftings of  $f$ , then  $F = F' + m$  with  $m \in \mathbb{Z}$ . Since the degree of  $f$  is one, we have  $F(X + 1) = F(X) + 1$  for all  $X \in \mathbb{R}$ . For  $x \in \mathbb{S}^1$ , the  $\limsup_{n \rightarrow \infty} (F^n(X) - X)/n$  exists for all  $X \in e^{-1}(x)$  and is independent of  $X$  (Newhouse, Palis and Takens [45]). We shall call this limit the rotation number  $\rho_F(x)$  of  $x$ . We denote by  $R(f)$  the set of all rotation numbers of  $f$ . From Misiurewicz [41] and Ito [27] we know that  $R(f)$  is a non-empty closed interval on  $\mathbb{R}$  (sometimes degenerated to one point) and, from now on, we shall call it the rotation interval of  $f$ .

Let  $f \in \mathcal{S}$  have degree one and rotation interval  $[c, d]$ . For  $c < d$  and  $t > 1$ , we define

$$R_{c,d}(t) = \sum t^{-q},$$

where the sum is taken over all pairs of integers  $(p, q)$  such that  $q > 0$  and  $c < p/q < d$  not necessarily coprime. Let  $\beta_{c,d}$  be the largest root of the equation  $R_{c,d}(t) = 1/2$ .

**Theorem 15.** *If the circle map  $f$  of degree 1 has rotation interval  $[c, d]$  with  $c < d$ , then  $h(f) \geq \log \beta_{c,d}$ . Moreover, for every pair  $c, d$  with  $c < d$  there is a circle map  $f$  of degree 1 with a rotation interval  $[c, d]$  and topological entropy  $\log \beta_{c,d}$ .*

Theorem 15 when  $c$  or  $d$  is zero is proved in [5], and in the general case in [3].

**Theorem 16.** *If the circle map  $f$  has degree  $d$ , then  $h(f) \geq \log |d|$ .*

Theorem 16 is proved by Block, Guckenheimer, Misiurewicz and Young in [8].

**Theorem 17.** *Let  $f$  be a transitive circle map. Then either  $h(f) > 0$  or  $f$  is conjugate to an irrational rotation (via a homeomorphism of degree 1).*

Theorem 17 is proved in [4].

**3.6. Entropy for graph maps.** A *graph*  $G$  is a connected compact space, which is the union of finitely many subsets homeomorphic to the interval  $[0, 1]$ , called *edges*, with pairwise disjoint interiors. The endpoints of the edges are called *vertices*. A continuous map  $f : G \rightarrow G$  is called a *graph map*.

**3.6.1. Entropy and periodic orbits.** In this subsection we relate the topological entropy of a graph map with its periodic orbits. A point  $x \in G$  is a *fixed point* for  $f$  if  $f(x) = x$ . A point  $x \in G$  is a *periodic point* of  $f$  of (least) period  $n$  if  $f^n(x) = x$  and  $f^k(x) \neq x$  for  $1 \leq k < n$ .

**Theorem 18.** *Let  $f$  be a graph map with positive topological entropy. Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n \geq h(f),$$

where  $\alpha_n$  denotes the number of periodic orbits of  $f$  of period  $n$ .

For  $k \in \mathbb{N}$  we denote by  $\text{god}(k)$  the *greatest odd divisor* of  $k$ . For a set  $S \subset \mathbb{N}$ , the set of gods of  $S$  (that is,  $\{\text{god}(k) : k \in S\}$ ) will be called the *pantheon* of  $S$  and  $\rho(S)$  will denote the *upper density* of  $S$  which is defined by

$$\rho(S) = \limsup_{n \rightarrow \infty} \frac{1}{n} \text{Card}\{k \in S : k \leq n\}.$$

Clearly, if  $S$  is finite then  $\rho(S) = 0$ . Also, for  $s \in \mathbb{N}$  we set

$$\Gamma_s = \prod_p E(\log(2s)/\log(p/2)) + 1,$$

where the product ranges over all odd prime numbers  $p \leq 4s$ .

We denote by  $\text{Per}(f)$  the set of all periods of the periodic points of a map  $f$ .

**Theorem 19.** *Let  $G$  be a graph and let  $f$  be a graph map. If  $G$  has  $s$  edges and the cardinality of the pantheon of  $\text{Per}(f)$  is larger than  $s\Gamma_s$  then  $h(f) > 0$ .*

The estimate of  $s\Gamma_s$  to assure positive topological entropy is not the best possible. Up to now the minimal number of gods that the set of periods of a map must have in order that it has positive topological entropy is unknown. But if the graph under consideration is the interval or the circle then this number is two.

**Theorem 20.** *Let  $f$  be a graph map. Then the following statements are equivalent:*

- (a)  $h(f) > 0$ .
- (b) *There is an  $m \in \mathbb{N}$  such that  $\{mn : n \in \mathbb{N}\} \subset \text{Per}(f)$ .*
- (c)  $\rho(\text{Per}(f)) > 0$ .
- (d) *The pantheon of  $\text{Per}(f)$  is infinite.*

Theorems 18 and 20 have been proved in [33]. Theorem 20 was actually proved by Blokh [12] by using the spectral decomposition for graph maps described in the same paper. The proof in [33] is more direct, it uses Theorem 5.

A *branching point* is a vertex which is the endpoint of at least three edges (if an edge has both endpoints at that vertex, we count the edge twice).

**Theorem 21.** *Let  $f$  be a graph map on the graph  $G$  with  $e$  endpoints,  $s$  edges,  $v$  vertexes and at least one branching point, which keeps all branching points fixed. Then  $h(f) > 0$  if and only if  $\text{god}(n) > e + 2s - 2v + 2$  for some period  $n$  of  $f$ .*

Theorem 21 is proved in [34].

**3.6.2. Transitive graph maps.** Let  $X$  be a metric space and let  $f : X \rightarrow X$  continuous. The map  $f$  will be called *totally transitive* if  $f^s$  is transitive for all integers  $s \geq 1$ .

Let  $(X, \mu)$  be a metric space (with more than one point). A continuous map  $f : X \rightarrow X$  has the *specification property* if for any

$\varepsilon > 0$  there exists  $M(\varepsilon) \in \mathbb{N}$  such that for any collection of  $k \geq 2$  points  $x_1, x_2, \dots, x_k \in X$ , for any collection of non-negative integers  $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$  such that  $a_i - b_{i-1} \geq M(\varepsilon)$  and for any  $p \in \mathbb{N}$  such that  $p \geq M(\varepsilon) + b_k - a_1$ , there exists a point  $y \in X$  such that  $f^p(y) = y$  and  $\mu(f^n(y), f^n(x_i)) \leq \varepsilon$  for all  $a_i \leq n \leq b_i$ ,  $1 \leq i \leq k$ . This means that if a map has the specification property, then any set of pieces of orbits can be approximated by one periodic orbit, provided that the times for “connections” between leaving one piece and coming close to the next one are sufficiently long.

The next theorem characterizes dynamically the transitive graph maps.

**Theorem 22.** *Let  $f$  be a graph map on the graph  $G$ . Then the following statements hold.*

- (a) *If  $f$  is transitive and  $\text{Per}(f)$  is empty, then  $G = \mathbb{S}^1$  and  $f$  is conjugate to an irrational rotation of the circle. Consequently,  $h(f) = 0$ .*
- (b) *If  $f$  is totally transitive and  $\text{Per}(f)$  is not empty, then  $f$  has the specification property. Consequently,  $h(f) > 0$ .*

Statement (a) of Theorem 22 was proved by Auslander and Katznelson [6] and Statement (b) by Blokh [11] and [10].

**3.6.3. Graph maps and Lefschetz numbers.** Let  $G$  be a graph. A *circuit* of  $G$  is a subset of  $G$  homeomorphic to a circle. Let  $f : G \rightarrow G$  be a graph map. The rational homology groups of a graph  $G$  are well-known. Thus we have that  $H_0(G; \mathbb{Q}) \approx \mathbb{Q}$  and  $H_1(G; \mathbb{Q}) \approx \mathbb{Q}^c$ , where  $c$  is the number of independent circuits of  $G$  in the sense of the homology. Let  $f_{*1} : H_1(G; \mathbb{Q}) \rightarrow H_1(G; \mathbb{Q})$  be the endomorphism induced by  $f$  on the first rational homology group of  $G$ . In fact,  $f_{*1}$  is a  $c \times c$  matrix with integer entries. Given a matrix  $A$  we denote its spectral radius by  $\text{sp}(A)$ , and its trace by  $\text{Tr}(A)$ .

The *Lefschetz number* of a graph map  $f$  is defined to be

$$L(f) = 1 - \text{Tr}(f_{*1}).$$

If  $L(f) \neq 0$  then  $f$  has a fixed point by the Lefschetz fixed point theorem.

The *Moebius function* is defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } k^2 \text{ divides } m \text{ for some } k \in \mathbb{N}, \\ (-1)^r & \text{if } m \text{ is a product of } r \text{ distinct prime factors.} \end{cases}$$



Let  $f$  be a graph map of the graph  $G$ . For every  $m \in \mathbb{N}$  we define the *Lefschetz number of period  $m$*  as follows

$$l(f^m) = \sum_{d \in \mathbb{N}, d|m} \mu(d) L(f^{m/d}).$$

Therefore

$$L(f^m) = \sum_{d \in \mathbb{N}, d|m} l(f^d).$$

The *asymptotic Lefschetz number*  $L^\infty(f)$  is defined to be the growth rate of the Lefschetz number of the iterates of  $f$ :

$$L^\infty(f) = \max \left\{ 1, \limsup_{m \rightarrow \infty} |L(f^m)|^{1/m} \right\}.$$

The asymptotic Lefschetz number allows to obtain a lower bound for the topological entropy of a continuous graph map.

**Theorem 23.** *Let  $f : G \rightarrow G$  be a graph map.*

- (a)  $L^\infty(f) = \max\{1, \text{sp}(|f_{*1}|)\}$ .
- (b) *The topological entropy of  $f$  satisfies  $h(f) \geq \log L^\infty(f)$ .*

Statement(a) of Theorem 23 is proved in [25], and statement (b) is due to Jiang [29, 28].

#### 4. PART II. TOPOLOGICAL ENTROPY IN SPACES OF DIMENSION $> 1$

**4.1. Entropy and volume growth.** Suppose that there is a Riemannian metric on the manifold  $M$ . If  $D \subset M$  is a  $C^1$  disk inside the manifold  $M$  and  $f$  is at least  $C^1$ , then the *volume growth of  $D$  under  $f$*  is

$$v(D, f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Vol}(f^n(D)),$$

where the volume  $\text{Vol}$  is with respect to the Riemannian metric on  $M$ . For  $1 \leq r \leq \infty$ , the supremum of all the volume growths over all the  $C^r$  disks in  $M$  is the  *$r$ -volume growth of  $f$* , i.e.

$$v_r(f) = \sup_{D \subset M \text{ } C^r \text{ disk}} v(D, f).$$

For  $1 \leq r_1 < r_2$  we clearly have that  $v_{r_1}(f) \geq v_{r_2}(f)$ .

For  $C^r$  maps, with  $r > 1$ , the volume growth is greater than the topological entropy.

**Theorem 24.** *If  $f$  is  $C^r$  on the compact Riemannian manifold  $M$ , with  $r > 1$ , then  $v_r(f) \geq h(f)$ .*

The proof of Theorem 24 is due to Newhouse [46], it is based on Pesin Theory, this is why the map is required to be  $C^r$ , with  $r > 1$ .

The volume growth can also be bounded from above in terms of the topological entropy.

**Theorem 25.** *Suppose that  $f$  is  $C^r$  on the compact Riemannian manifold  $M$ , with  $r \geq 1$ . Denote by*

$$R(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\sup_{x \in M} \|df^n x\|).$$

*Then  $v_r(f) \leq h(f) + \frac{m}{r} R(f)$ .*

Theorem 25 is due to Yomdin, see [55], [56] and [23].

**Corollary 26.** *If  $f$  is  $C^\infty$  on the compact Riemannian manifold  $M$ , then  $v_\infty(f) = h(f)$ .*

This corollary is proved by Yomdin in [56].

**4.2. Entropy and periodic points.** Let  $M$  be an  $m$ -dimensional compact connected Riemannian manifold and  $f : M \rightarrow M$  a continuous map. We say that  $f$  satisfies the *hypothesis* (H1) if the fixed points of  $f^n$  are isolated for all positive integers  $n$ .

Hypothesis (H1) holds for  $C^r$  generic maps,  $r \geq 1$ . Here we say that a property is  $C^r$  generic,  $r \geq 0$ , if the property holds for a residual subset of  $C^r$  maps, considered with the  $C^r$  topology.

If  $f$  satisfies the hypothesis (H1), then we denote by  $\text{CardFix}(f^n)$  the number of fixed points of  $f^n$  (this is the number of periodic points of periods divisors of  $n$ ). Then one can define the *rate of growth of periodic points* to be the rate of growth of these numbers with respect to  $n$ , i.e.

$$\text{Per}^\infty(f) = \limsup_{n \rightarrow \infty} (\max\{\text{CardFix}(f^n), 1\})^{\frac{1}{n}}.$$

The rate of growth of periodic points is again a bit more complicated, and it cannot be related in general to the topological entropy and the volume growth. For a  $C^1$  map the volume growth is always finite, and for a Lipschitz map the topological entropy is finite, but the rate of growth of periodic points may be infinite for  $C^r$  maps, with  $r \geq 2$ .

**Theorem 27.** *There exist an open set of  $C^r$  diffeomorphisms, with  $r \geq 2$ , which contains a residual set of maps with super-exponential growth of periodic points, so in this case  $\text{Per}^\infty(f) = \infty$ .*

Theorem 27 is due to Kaloshin [30].

For Axiom A diffeomorphisms Bowen in [13] proved that the entropy is equal to the logarithm of the rate of growth of periodic points.

**Theorem 28.** *If  $f$  is an Axiom A diffeomorphism on the compact manifold  $M$ , then  $h(f) = \log \text{Per}^\infty(f)$ .*

**4.3. Entropy conjecture.** The map  $f$  induces an action on the homology groups of  $M$ , which we denote  $f_{*,k} : H_k(M, \mathbb{Q}) \rightarrow H_k(M, \mathbb{Q})$ , for  $k \in \{0, 1, \dots, m\}$ . The *spectral radii* of these maps are denoted  $\text{sp}(f_{*,k})$  and they are equal to the largest modulus of all the eigenvalues of the linear map  $f_{*,k}$ . The *spectral radius of  $f_*$*  is

$$\text{sp}(f_*) = \max_{k=0, \dots, m} \text{sp}(f_{*,k}).$$

If we assume that  $M$  is oriented, then the top homology group  $H_m(M, \mathbb{Q})$  is homeomorphic to  $\mathbb{Q}$  and  $f_{*,m}$  is just the multiplication by an integer  $\deg(f)$  which is called the *degree* of  $f$ .

**Conjecture 29** (Entropy conjecture). *If  $f$  is a  $C^1$  map on a compact manifold  $M$ , then the topological entropy is greater than or equal to the logarithm of the spectral radius of  $f$ :*

$$(1) \quad \log(\text{sp}(f_*)) \leq h(f).$$

This conjecture is due to Shub [52], see also [31], and [20] for a slight generalization.

Inequality (1) is not true for Lipschitz maps, see [52] and [23].

The entropy conjecture seems very difficult, but there are some partial results. Thus a weaker version of the conjecture is known to be true if we add a smoothness assumption on  $f$ , we require it to be  $C^\infty$ .

**Theorem 30.** *If  $f$  is  $C^\infty$  on the compact manifold  $M$ , then the entropy conjecture is true, i.e.  $\log(\text{sp}(f_*)) \leq h(f)$ .*

Theorem 30 is due to Yomdin [55].

Some weaker versions of the entropy conjecture are obtained by replacing the spectral radius of  $f$  by some of the other global invariants mentioned in Section 3. For example the logarithm of the degree is smaller than or equal to the topological entropy for  $C^1$  maps.

**Theorem 31.** *If  $f$  is  $C^1$  on the compact oriented manifold  $M$ , then  $\log(\deg(f)) \leq h(f)$ .*

Theorem 31 is due to Misiurewicz and Przytycki [43].

Again Theorem 31 is not true for Lipschitz maps, see for instance [34].

**Theorem 32.** *If  $f$  is  $C^0$  and  $M$  is a compact manifold, then  $\log(\text{sp}(f_{*,1})) \leq h(f)$ .*

Theorem 32 is due to Manning [35].

Fix  $p \in M$  and a path  $\alpha$  joining  $p$  with  $f(p)$ . We denote by  $\pi_1(M, p)$  the fundamental group of the space  $M$  at the point  $p$ . Define the endomorphism  $f_*^\alpha : \pi = \pi_1(M, p) \rightarrow \pi$  by  $f_*^\alpha(\gamma) = \alpha f(\gamma) \alpha^{-1}$ . Let  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_s\}$  be a set of generators of  $\pi$  and define the length of an element  $\gamma \in \pi$  as

$$L(\gamma, \Gamma) = \min \left\{ \sum_{j=1}^l |i_j| : \gamma = \gamma_{s_1}^{i_1} \gamma_{s_2}^{i_2} \dots \gamma_{s_l}^{i_l}, l \geq 1, 1 \leq s_1, \dots, s_l \leq s \right\}.$$

The *fundamental-group entropy* of  $f$  is

$$h_*(f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \max_{1 \leq i \leq s} L((f_*^\alpha)^n(\gamma_i), \Gamma) \right).$$

It can be proved that  $h_*(f)$  is well defined and independent of  $\Gamma$ ,  $p$  and  $\alpha$ . For more about this see for example [26] or [16] (in Bowen's paper the fundamental-group entropy is called the logarithm of the *growth rate of  $f_*$  on  $\pi_1(M)$* ).

Bowen in [16] extended the result of Manning to the fundamental group of  $M$ :

**Theorem 33.** *If  $f$  is  $C^0$  and  $M$  is a compact manifold, then  $h_*(f) \leq h(f)$ .*

Katok in [31] proposed another version for the entropy conjecture:

**Conjecture 34.** *If  $f$  is a continuous self-map on a compact manifold  $M$  with the universal cover homeomorphic to an Euclidean space, then  $\log(\text{sp}(f_*)) \leq h(f)$ .*

On one hand this is weaker than Shub's entropy conjecture because there are some restrictions on the manifold  $M$ , but on the other hand it is stronger because it only requires that  $f$  is continuous.

In [36] Marzantowicz and Przytycki showed that the entropy conjecture is true for continuous self-maps on nilmanifolds, and in [37] they obtained the following generalization:

**Theorem 35.** *If  $f$  is a continuous self-map of a compact  $K(\pi, 1)$  manifold  $M$ , with the fundamental group  $\pi$  torsion free and virtually nilpotent, then  $\log(\text{sp}(f_*)) \leq h(f)$ .*

Another way to obtain versions of the entropy conjecture is to add some restrictions on the dynamics of  $f$ . Shub and Williams [54] and Ruelle and Sullivan [50] proved that the entropy conjecture is true for Axiom A plus no-cycle condition diffeomorphisms:

**Theorem 36.** *If  $f$  is a diffeomorphism on  $M$  which satisfies Axiom A and the no-cycle condition, then  $\log(\text{sp}(f_*)) \leq h(f)$ .*

The entropy conjecture holds for partially hyperbolic diffeomorphisms with one-dimensional center, see Saghin and Xia [51]:

**Theorem 37.** *If  $f$  is a partially hyperbolic diffeomorphism on the compact manifold  $M$  and the center bundle of  $f$  is one-dimensional, then  $\log(\text{sp}(f_*)) \leq h(f)$ .*

The entropy conjecture holds also when the map  $f$  is  $C^1$  and it has a finite chain-recurrent set, see Fried and Shub [21]:

**Theorem 38.** *If  $f$  is a  $C^1$  diffeomorphism on the compact manifold  $M$  and the chain-recurrent set of  $f$  is finite, then  $\log(\text{sp}(f_*)) \leq h(f)$ .*

**4.4. Volume growth and the spectral radius.** The relation between the topological entropy and the volume growth was studied in section 4.1. In section 4.3 we considered the relation between the topological entropy and the spectral radius of the homology. The next proposition shows the well known fact, see for instance [55], that for a  $C^1$  map the volume growth is greater than or equal to the logarithm of the spectral radius, and in particular of the logarithm of the degree.

**Proposition 39.** *If  $f$  is  $C^1$  on the compact manifold  $M$ , then for all  $1 \leq r \leq \infty$  we have  $\log(\text{sp}(f_*)) \leq v_r(f)$ , and consequently  $\log(\deg(f)) \leq v_r(f)$ .*

#### ACKNOWLEDGMENTS

We thank to the referee their comments which helped to improve the presentation of this paper.

The author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014SGR-568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

#### REFERENCES

- [1] R.L. ADLER, A.G. KONHEIM AND M.H. MCANDREW, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.
- [2] R.L. ADLER AND M.H. MCANDREW, *The entropy of Chebyshev polynomials*, Trans. Amer. Math. Soc. **121** (1966), 236–241.
- [3] L. ALSEDÀ, J. LLIBRE, F. MAÑOSAS AND M. MISIUREWICZ, *Lower bounds of the topological entropy for continuous maps of the circle of degree one*, Nonlinearity **1** (1988), 463–479.

- [4] L. ALSÈDÀ, J. LLIBRE AND M. MISIUREWICZ, *Combinatorial dynamics and entropy in dimension one*, Second edition, Advanced Series in Nonlinear Dynamics Vol. **5**, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
- [5] L. ALSÈDÀ, J. LLIBRE, M. MISIUREWICZ AND C. SIMÓ, *Twist periodic orbits and topological entropy for continuous maps of the circle of degree one which have a fixed point*, Ergod. Th. & Dynam. Sys. **5** (1985), 501–517.
- [6] J. AUSLANDER AND Y. KATZNELSON, *Continuous maps of the circle without periodic points*, Israel J. Math. **32** (1979), 375–381.
- [7] L. BLOCK AND E.M. COVEN, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. **300** (1987), 297–306.
- [8] L. BLOCK, J. GUCKENHEIMER, M. MISIUREWICZ AND L.-S. YOUNG, *Periodic points and topological entropy of one dimensional maps*, Global theory of dynamical systems, 18–34, Lecture Notes in Math. **819**, Springer, Berlin, 1980.
- [9] A.M. BLOKH, *On sensitive mappings of the interval*, Russian Math. Surveys **37:2** (1982), 203–204.
- [10] A.M. BLOKH, *On transitive mappings of one-dimensional branched manifolds* (in Russian), Diff. -Difference Equations and Problems of Mathematical Physics, Inst. of Math., Kiev (1984), 3–9.
- [11] A.M. BLOKH, *On the connection between entropy and transitivity for one-dimensional mappings*, Russian Math. Surveys **42** (1987), 165–166.
- [12] A.M. BLOKH, *The Spectral Decomposition, Periods of Cycles and Misiurewicz Conjecture for Graph Maps*, 24–31, Lecture Notes in Math., Springer **1514**, 1991.
- [13] R. BOWEN, *Topological entropy and axiom A*, Proc. Symp. in Pure Math. AMS vol 14, Global Analysis (1970), 23–41.
- [14] R. BOWEN, *Entropy for group endomorphisms and homogeneous spaces* Trans. Amer. Math. Soc. **153** (1971), 401–414; erratum: Trans. Amer. Math. Soc. **181** (1973), 509–510.
- [15] R. BOWEN, *Entropy for maps of the interval*, Topology **16** (1977), 465–467.
- [16] R. BOWEN, *Entropy and the fundamental group*, in The structure of attractors in dynamical systems, Lecture Notes in Math. **668**, Springer-Verlag, Berlin 1978, 21–29.
- [17] P. BOYLAND, *Topological methods in surface dynamics*, Topology Appl. **58** (1994), 223–298.
- [18] P. BOYLAND, *Isotopy stability of dynamics on surfaces*, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999), 17–45, Contemp. Math. **246**, Amer. Math. Soc., Providence, RI, 1999.
- [19] J. FRANKS AND M. MISIUREWICZ, *Topological methods in dynamics*, Handbook of dynamical systems, Vol. 1A, 547–598, North-Holland, Amsterdam, 2002.
- [20] D. FRIED, *Entropy and twisted cohomology*, Topology **25** (1986), 455–470.
- [21] D. FRIED AND M. SHUB, *Entropy linearity and chain-recurrence*, Publ. Math. de l’IHES **50** (1979), 203–214.
- [22] S. FRIEDLAND, *Entropy of holomorphic and rational maps: a survey*, preprint Arxiv.

- [23] M. GROMOV, *Entropy, homology and semialgebraic geometry (after Y. Yomdin)*, Seminaire N. Bourbaki, Volume 1985-1986, Exposé 663.
- [24] M. GROMOV, *Three remarks on the geodesic dynamics and fundamental groups*, L'Enseign. Math. **46** (2000), 391–402.
- [25] J. GUASCHI AND J. LLIBRE, *Periodic points of  $C^1$  maps and the asymptotic Lefschetz number*, Inter. J. of Bifurcation and Chaos **5** (1995), 1369–1373.
- [26] B. HASSELBLATT AND A. KATOK, *Handbook of dynamical systems*, Vol. 1A. North-Holland, Amsterdam, 2002.
- [27] R. ITO, *Rotation sets are closed*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 107–111.
- [28] B. JIANG, *Nielsen theory for periodic orbits and applications to dynamical systems*, Contemp. Math. **152** (1993), 183–202.
- [29] B. JIANG, *Estimation of the number of periodic orbits*, Pacific J. of Math. **172** (1996), 151–185.
- [30] V. Y. KALOSHIN, *Generic diffeomorphisms with superexponential growth of number of periodic orbits*, Comm. Math. Phys. **211** (2000), 253–271.
- [31] A. KATOK, *The entropy conjecture*, in Smooth dynamical systems (Russian), pp. 181–203. Izdat. "Mir", Moscow, 1977.
- [32] A. KATOK, *Fifty years of entropy in dynamics: 1958–2007*, J. Mod. Dyn. **1** (2007), no. 4, 545–596.
- [33] J. LLIBRE AND M. MISIUREWICZ, *Horseshoes, entropy and periods for graph maps*, Topology **52** (1993), 149–164.
- [34] J. LLIBRE, R. SAGHIN, *Results and open questions on some of the invariants measuring the dynamical complexity of a map*, Fundamenta Mathematica (Proc. of Michal Misiurewicz) **206** (2009), 307–327.
- [35] A. MANNING, *Topological entropy and the first homology group*, in Dynamical systems - Warwick 1974, Lecture Notes in Math **468**, Springer-Verlag, Berlin, 1975, 185–190.
- [36] W. MARZANTOWICZ AND F. PRZYTYCKI, *Entropy conjecture for continuous maps of nilmanifolds*, Israel J. of Math. **165**, (2008), 349–379.
- [37] W. MARZANTOWICZ AND F. PRZYTYCKI, *Estimates of the topological entropy from below for continuous self-maps on some compact manifolds*, Discrete Contin. Dyn. Syst.- Series A **21**, (2008), 501–512.
- [38] J. MILNOR AND W. THURSTON, *On iterated maps of the interval*, Dynamical systems, 465–563, Lecture Notes in Math. **1342**, Springer, Berlin, 1988.
- [39] M. MISIUREWICZ, *Horseshoes for mappings of an interval*, Bull. Acad. Pol. Sci., Sér. Sci. Math. **27** (1979), 167–169.
- [40] M. MISIUREWICZ, *Horseshoes for continuous mappings of an interval*, Dynamical systems, 127–135, Liguori, Napoli, 1980.
- [41] M. MISIUREWICZ, *Twist sets for maps of the circle*, Ergodic Theory & Dynam. Systems **4** (1984), 391–404.
- [42] M. MISIUREWICZ, *Jumps of entropy in one dimension*, Fund. Math. **132** (1989), 215–226.
- [43] M. MISIUREWICZ AND F. PRZYTYCKI, *Topological entropy and degree of smooth mappings*, Bull. Ac. Pol. Sci. **25** (1997), 573–574.
- [44] M. MISIUREWICZ AND W. SZLENK, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.

- [45] S. NEWHOUSE, J. PALIS AND F. TAKENS, *Bifurcations and stability of families of diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **57** (1983), 5–71.
- [46] S. NEWHOUSE, *Entropy and volume*, Erg. Th. & Dyn. Syst. **8** (1988), 283–299.
- [47] S. NEWHOUSE, *Entropy in smooth dynamical systems*, Proceedings of the International Congress of Mathematicians Vol. I, II (Kyoto, 1990), 1285–1294, Math. Soc. Japan, Tokyo, 1991.
- [48] W. PARRY, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. **122** (1966), 368–378.
- [49] J. ROTHCHILD, *On the computation of topological entropy*, Thesis, CUNY, 1971.
- [50] D. RUELLE AND D. SULLIVAN, *Currents, flows and diffeomorphisms*, Topology **14** (1975), 319–327.
- [51] R. SAGHIN AND Z. XIA, *The entropy conjecture for partially hyperbolic diffeomorphism with 1-D center*, Topology Appl. **157** (2010), 29–34.
- [52] M. SHUB, *Dynamical Systems, filtrations and entropy*, Bull. Amer. Math. Soc. **80** (1974), 27–41.
- [53] M. SHUB, *All, most, some differentiable dynamical systems*, International Congress of Mathematicians, Vol. III, 99–120, Eur. Math. Soc., Zurich, 2006.
- [54] M. SHUB AND R. WILLIAMS, *Entropy and Stability*, Topology **14** (1975), 329–338.
- [55] Y. YOMDIN, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.
- [56] Y. YOMDIN,  *$C^k$ -resolution of semialgebraic mappings*, Addendum to Volume growth and entropy, Israel J. Math. **57** (1987), 301–317.
- [57] L.S. YOUNG, *Entropy in dynamical systems*, Entropy 313–327, Princeton Ser. Appl. Math., Princeton Univ. Press, Princeton, NJ, 2003.
- [58] L.S. YOUNG, *On the prevalence of horseshoes*, Trans. Amer. Math. Soc. **263** (1981), 75–88.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA,  
 BELLATERRA, 08193, BARCELONA, CATALONIA, SPAIN  
*E-mail address:* jllibre@mat.uab.cat