ON THE CENTRAL CONFIGURATIONS OF THE n–BODY PROBLEM

JAUME LLIBRE

Abstract. We present a brief survey on some classes of central configurations of the n–body problem. We put special emphasis on the central configurations of the $1+n$–body problem also called the coorbital satellite problem, and on the nested central configurations formed by either regular n–gons, or regular polyhedra. We also present some conjectures.

1. Introduction

1.1. **The n–body problem.** The *n–body problem* consists in studying the motion of n pointlike masses, interacting among themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law.

The *equations of motion* of the n–body problem are

$$m_k \mathbf{r}''_k = \sum_{j=1, j\neq i}^{n} \frac{G m_j m_k}{r_{jk}^3} (\mathbf{r}_j - \mathbf{r}_k),$$

for $k = 1, \ldots, N$, where G is the gravitational constant, $\mathbf{r}_k \in \mathbb{R}^3$ is the position vector of the punctual mass m_k in an inertial system, and r_{jk} is the Euclidean distance between the masses m_j and m_k.

The *center of mass* of the system formed by the n bodies satisfies

$$\frac{\displaystyle\sum_{k=1}^{N} m_k \mathbf{r}_k}{m_1 + \ldots + m_N} = \mathbf{a}t + \mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. Without loss of generality we can consider that the center of mass is at the origin of the inertial system, i.e.

$$\sum_{k=1}^{N} m_k \mathbf{r}_k = \mathbf{0}.$$

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A such inertial system is called *inertial barycentric system*. In the rest of this paper we will work in an inertial barycentric system.

1.2. **Homographic solutions of the n–body problem.** Since the general solution of the n–body problem cannot be given, from the very beginning great importance has been dedicated to the search for particular solutions where the n mass points fulfilled certain initial conditions.

A *homographic solution* of the n–body problem is a solution such that the configuration formed by the n–bodies at the instant t (with respect to an inertial barycentric system) remains similar to itself as t varies.

Two configurations are *similar* if we can pass from one to the other doing a dilatation and/or a rotation.

The first three homographic solutions where found in 1767 by Euler [16] in the 3–body problem. For these three solutions the configuration of the three bodies is collinear.

In 1772 Lagrange [23] found two additional homographic solutions in the 3–body problem, the configurations formed by the three bodies at the vertices of an equilateral triangle.

1.3. **Central configurations of the n–body problem.** At a given instant $t = t_0$ the configuration of the n–bodies is *central* if the gravitational acceleration $\mathbf{r}''_k$ acting on every mass point m_k is proportional with the same constant of proportionality to its position $\mathbf{r}_k$ (referred to an inertial barycentric system); i.e.

$$\mathbf{r}''_k = \sum_{j=1, j\neq k}^{n} \frac{Gm_j}{r_{jk}^3}(\mathbf{r}_j - \mathbf{r}_k) = \lambda \mathbf{r}_k, \qquad \text{for } k = 1, \ldots, N.$$

Pizzetti [36] in 1904 proved that the configuration of the n bodies in a homographic solution is central at any instant of time.

It is important to note that homographic solutions with rotation and eventually with a dilatation only exist for planar central configurations. For spatial central configurations all the homographic solutions only have a dilation, for more details see Wintner [47].

For additional information on the central configurations of the n–body problem see for instance Albouy and Chenciner [1], Dziobek [15], Hagihara [20], Llibre [24], Moeckel [31, 26], Palmore [34], Saari [38, 39], Schmidt [42], Wintner [47] and Xia [48], ...

1.4. **Importance of the central configurations.** Central configurations of the n–body problem are important because:

(1) They allow to compute all the homographic solutions, see [47].
(2) If the n bodies are going to a simultaneous collision, then the particles tend to a central configuration, see [14].
(3) If the n bodies are going simultaneously at infinity in parabolic motion (i.e. the radial velocity of each particle tends to zero as the particle tends to infinity), then the particles tend to a central configuration, see [40].
(4) There is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum, see [45, 46].
(5) The Trojan asteroids are around one vertex of the equilateral triangle having at the other two vertices the Sun and Jupiter, see [37].
(6) Central configurations provides good places for the observation in the solar system, for instance see the SOHO project [18, 19].
(7) ...

1.5. **Classes of central configurations.** If we have a central configuration, a dilatation and a rotation (centered at the center of masses) of it, provides another central configuration. We say that two central configurations are related if we can pass from one to another through a dilation and a rotation. This relation is an equivalence. In what follows we will talk about the classes of central configurations defined by this equivalent relation.

1.6. **Collinear central configurations.** In 1910 Moulton [33] characterized the number of (classes) collinear central configurations by showing that there exist exactly $n!/2$ classes of collinear central configurations of the n–body problem for a given set of positive masses, one for each possible ordering of the particles modulo a rotation of π radians.

1.7. **Planar and spatial central configurations.** In the rest of this paper we are only interested either in planar central configurations which are not collinear, or spatial central configurations which are not planar.

For arbitrary masses $m_1, \ldots, m_n$ the planar central configurations are in general unknown when the number of the bodies $n > 3$. Numerically for $n = 4$ are known, see Simó [43].

For arbitrary masses $m_1, \ldots, m_N$ the spatial central configurations are in general unknown when the number of the bodies $n > 4$. We note that for $n = 4$ and arbitrary four masses at the vertices of a regular tetrahedron, we have the unique spatial central configuration for $n = 4$.

Wintner [47] in 1941 asked: Are there finitely many classes of planar central configurations for any choice of the masses $m_1, \ldots, m_n$ when $n > 3$?

This question also appears in the Smale's list on the mathematical problems for the XXI century, see [44].

In 2006 and for $n = 4$ Hampton and Moeckel [22] provided a positive answers to Wintner's question given an assisted proof by computer. Later on in 2012 Albouy and Kaloshin [3] gave an analytic proof for $n = 4$ and almost a proof for $n = 5$.

2. Central configurations of the coorbital satellite problem

Now we want to consider a restricted version of the planar central configurations; i.e. we study the limit case of one large mass and n small equal masses when the small masses tend to zero.

This problem is called *the central configurations of the planar* $1+n$*–body problem*, or *the central configurations of the coorbital satellite problem.*

The $(1+n)$–body problem was first considered by Maxwell [30] when he tried to explain the stability of the motion of Saturn's Rings in 1885.

In the $(1+n)$–body problem the infinitesimal particles interact between them under the gravitational forces, but they do not perturb the largest mass.

Let $\mathbf{r}(\varepsilon) = (\mathbf{r}_0(\varepsilon), \mathbf{r}_1(\varepsilon), \ldots, \mathbf{r}_n(\varepsilon))$ be a planar central configuration of the $(1+n)$–body problem with masses $m_0 = 1$ and $m_1 = \ldots = m_n = \varepsilon$, which depend continuously on ε when $\varepsilon \to 0$.

We say that $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_n)$ with $\mathbf{r}_i \neq \mathbf{r}_j$ if $i \neq j$ and $i, j \geq 1$, is a *central configuration of the planar* $(1+n)$*–body problem* if there exists the $\lim_{\varepsilon \to 0} \mathbf{r}(\varepsilon)$ and this limit is equal to $\mathbf{r}$.

From this definition it is clear that if we know the central configurations of the $(1+n)$–body problem, then we can continue them to sufficiently small positive values of ε.

Proposition 1. *The n infinitesimal bodies of a central configurations of the planar $(1+n)$–body problem lie on a circle $\mathbb{S}^1$ centered in the big body, i.e. at $\mathbf{r}_0$.*

Proposition 1 provides the reason for which the central configurations of the $(1+n)$–body problem have applications to the dynamics of the coorbital satellite systems.

Proposition 2. *Let $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_n)$ be a central configuration of the planar $(1+n)$–body problem. Denoting by θ_i the angle defined by the position of $\mathbf{r}_i$ on the circle $\mathbb{S}^1$, we have*

$$\sum_{j=1, j\neq k}^{n} \sin(\theta_j - \theta_k) \left[1 - \frac{1}{8|\sin^3(\theta_j - \theta_k)/2|}\right] = 0,$$

for $k = 1, \ldots, n$.

A proof of both propositions was given by Hall [21], see also [5].

Numerical results due to Salo and Yoder [41] provided the number of classes of central configurations described in the next table. For the n's of the table they also study the linear stability of the central configurations. But the linear stability of the central configurations of the $(1+n)$–body problem for all n was studied by Moeckel [32].

n	Number of central configurations
2	2
3	3
4	3
5	3
6	3
7	5
8	3
9	1

On the other hand numerical computations (see [13]) seem to indicate that

(i) for $n \geq 9$ the number of central configurations is 1 (we know numerically that this is the case for $9 \leq n \leq 100$), and
(ii) that every known central configuration of the $(1+n)$–body problem has a straight line of symmetry.

The numerical results of the table for $n = 2, 3, 4$, have been proved analytically by

- Euler and Lagrange for $n = 2$,
- Hall for $n = 3$,
- Cors, Llibre and Ollé for $n = 4$, and
- Albouy and Fu [2] proved the existence of the line of symmetry for $n = 4$.

For $n \geq 2$ it is known that the regular n–gon, having the infinitesimal particles in its vertices and with the large mass in its center, is a central configuration, see for instance [21, 5].

Additionally for $n \geq e^{27000}$ the regular n–gon is the unique central configuration, this was proved by Hall in 1988 in the unpublished paper [21]. Later on in 1994 it was proved that for $n \geq e^{73}$ the regular n–gon is the unique central configuration, see [5].

Conjectures. *For the central configurations of the* $(1+n)$*–body problem:*

(I) *the regular n–gon is the unique central configuration when $n \geq 9$, and*

(II) *every central configuration has a straight line of symmetry.*

Note that we only need to prove conjecture (II) for $5 \leq n \leq 8$, when conjecture (I) be proved for $9 \leq n \leq e^{73}$.

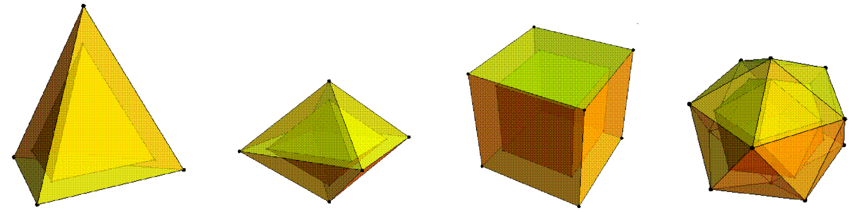
3. Planar nested central configurations

We do not know who was the first in proving that the regular n–gon with equal masses is a central configuration, but clearly in 1907 this result was well known, see for instance the paper of LongLey [28].

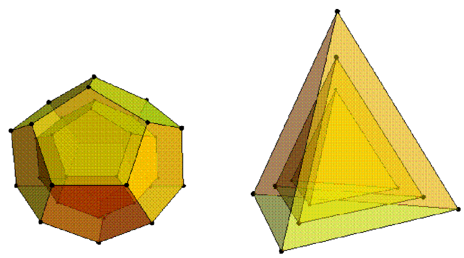
For $p = 2$ and $n = 2, 3, 4, 5, 6$ in [28], and for $p = 2$ and $n \geq 2$ Zhang and Zhou [49] proved that there are central configurations of the $2n$–body problem where n bodies with equal mass are in the vertices of a regular n–gon, and the remainder n bodies with equal mass but not necessarily equal to the previous n bodies, are also in the vertices of another regular n–gon homothetic to the first n–gon.

These previous results where extended to p nested regular n–gons with $p = 3, 4$ and $n \geq 2, 3, 4$ by Llibre and Mello in [25]. Finally Corbera, Delgado and Llibre [7] proved the following result.

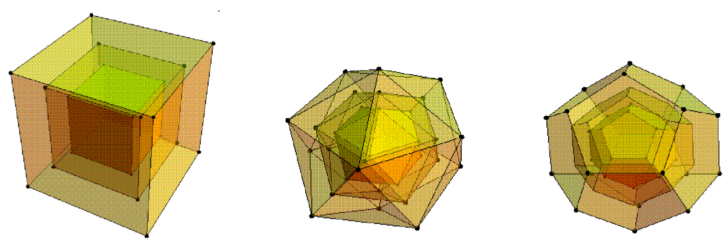
Theorem 3. *For all $p \geqslant 2$ and $n \geqslant 2$, we prove the existence of central configurations of the pn–body problem where the masses are at the vertices of p nested regular n–gons with a common center. In such configurations all the masses on the same n–gon are equal, but masses on different n–gons could be different.*



(a) Two tetraedra (b) Two octahedra (c) Two cube (d) Two icosahedra



(e) Two dodecahedra (f) Three tetrahedra (g) Three octahedra



(h) Three regular cube (i) Three icosahedra (j) Three dodecahedra

FIGURE 1. Nested regular polyhedra, with two or three polyhedra.

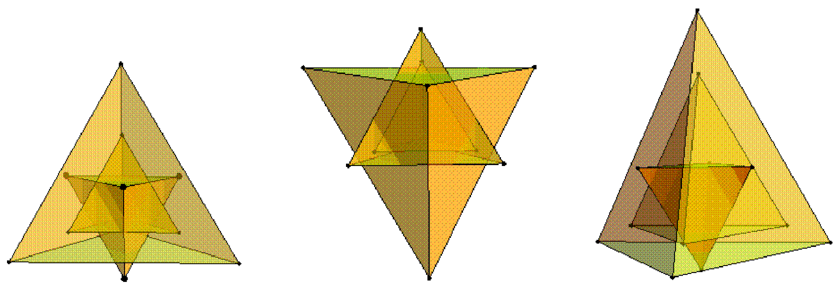
4. SPATIAL NESTED CENTRAL CONFIGURATIONS

Cedó and Llibre in [6] proved that if we put equal masses at the vertices of a regular polyhedra we get a central configuration. Later on Corbera and Llibre [8] proved the next result.

Theorem 4. *We consider* $2n$ *masses located at the vertices of two nested regular polyhedra with the same number of vertices. Assuming that the masses in each polyhedron are equal, we prove that for each ratio of the masses of the inner and the outer polyhedron there exists a unique ratio of the length of the edges of the inner and the outer polyhedron such that the configuration is central. See Figure 1.*

Theorem 4 was extended to 3 nested regular polyhedra in [9], see again Figure 1.

These results were extended to $p \geqslant 2$ nested polyhedra, see [11].



(a) (b) (c)

FIGURE 2. Three rotated tetrahedra.

Theorem 5. *For all $p \geqslant 2$ we prove the existence of central configurations of the pn–body problem where the masses are located at the vertices of p nested regular polyhedra having the same number of vertices n and a common center. In such configurations all the masses on the same polyhedron are equal, but masses on different polyhedra could be different.*

Later on these kind of nested central configurations with polyhedra were extended to rotated $p = 3$ nested regular polyhedra in [10], see Figure 2.

5. On the central configurations of the 4–body problem

The planar central configurations of the 4–body problem are classified as *convex* or *concave*. Thus a central configuration is *convex* if none of the bodies is located in the interior of the triangle formed by the other three. A central configuration is *concave* if one of the bodies is in the interior of the triangle formed by the other three.

From the paper of MacMillan and Bartky [29] we have the following two conjectures, see also Albouy and Fu [2] and Pérez-Chavela and Santoprete [35].

Conjecture (III). *The 4–body problem has a unique class of convex central configuration.*

Conjecture (IV). *If the 4–body problem has a central configurations with two pairs of equal masses located at two adjacent vertices of a convex quadrilateral, then this quadrilateral is an isosceles trapezoid.*

Long and Sun in [27] proved that any convex central configuration with masses $m_1 = m_2 < m_3 = m_4$ at the opposite vertices of a quadrilateral and having the diagonal corresponding to the mass m_1 longer than the one corresponding to the mass m_3, is a rhombus. Pérez–Chavela [35] and Santoprete extended this result to the case where two of the masses are equal and at most, only one of the remaining mass is larger than the equal masses. Furthermore, they shown that there exists only one convex central configuration if the opposite masses are equal and it is a rhombus. Albouy, Fu and Sun in [4] proved for the 4–body problem that a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. When these two masses are not equal, then the smallest mass is closer to the former diagonal.

Using these results on the symmetries Corbera and Llibre [12] proved Conjectures (III) and (IV) when two pairs of the masses are equal and one of the pairs of equals masses is sufficiently small.

Conjecture (IV) has been proved recently by Fernandes, Llibre and Mello in [17].

Conjecture (III) remains open for arbitrary values of the four masses.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat