THE LIMIT DYNAMICS FOR THE VACUUM EINSTEIN EQUATIONS IN A HOMOGENEOUS UNIVERSE

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ABSTRACT. We study the dynamics of the Bianchi IX universe when one of the structure constants tends to zero, i.e. we study the dynamics of the Bianchi VII universe. We prove that there is a surface filled of periodic orbits surrounding an equilibrium point, and that except for another surface filled of equilibria the remainder orbits comes and go to infinity.

1. Introduction and stament of the main results

Bianchi universes are homogeneous cosmlogical models non-necessarily isotropic in 3-dimensional spaces created by Luigi Bianchi [2]. Here I study the qualitative dynamics of the model Bianchi VII, who can be obtained as the limit of Bianchi IX when one of the structure constants tends to zero.

The vacuum Einstein equations for a homogeneous universe are

(1) e1
$$
\begin{aligned}
(\ln a^2)'' &= (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4,\\
(\ln b^2)'' &= (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4,\\
(\ln c^2)'' &= (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4,
\end{aligned}
$$

satisfying that

(2) ?e2?
$$
XY + XZ + YZ - \left(\lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 - 2\left(\lambda\mu a^2 b^2 + \mu\nu b^2 c^2 + \lambda\nu a^2 c^2\right)\right) = 0,
$$

where $X = (\ln a^2)'$, $Y = (\ln b^2)'$ and $Z = (\ln c^2)'$, see equations (22)–(24) of [3], and for more details [3]. As usual the prime denotes derivative with respect to the time t.

In order to study the dynamics of the differential system (1) of three second order differential equations we write this system as the following differential system of six first order differential equations

(3) e3
$$
\begin{aligned}
x' &= X,\\
X' &= (\mu e^y - \nu e^z)^2 - \lambda^2 e^{2x},\\
y' &= Y,\\
Y' &= (\lambda e^x - \nu e^z)^2 - \mu^2 e^{2y},\\
z' &= Z,\\
Z' &= (\lambda e^x - \mu e^y)^2 - \nu^2 e^{2z},
\end{aligned}
$$

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where $x = \ln a^2$, $y = \ln b^2$ and $z = \ln c^2$. This differential system has the first integral

$$H = XY + XZ + YZ - \left(\lambda^2 e^{2x} + \mu^2 e^{2y} + \nu^2 e^{2z} - 2\left(\lambda\mu e^{x+y} + \mu\nu e^{y+z} + \lambda\nu e^{x+z}\right)\right),$$

because the function $H = H(x, X, y, Y, z, Z)$ is constant on the solutions of the differential system (3) because

$$\frac{dH}{dt} = \frac{\partial H}{dx}x' + \frac{\partial H}{dX}X' + \frac{\partial H}{dy}y' + \frac{\partial H}{dY}Y' + \frac{\partial H}{dz}z' + \frac{\partial H}{dZ}Z' = 0.$$

In the paper [3] the authors obtained numerically information about the dynamics of the differential system (1) when $\lambda \to 0$. Our objective is to study analytically the differential system (1) when $\lambda \to 0$ and to describe its dynamics.

The differential system (3) when $\lambda \to 0$ becomes

$$\begin{aligned} &x' = X,\\ &X' = (\mu e^y - \nu e^z)^2,\\ &y' = Y,\\ &Y' = \nu^2 e^{2z} - \mu^2 e^{2y},\\ &z' = Z,\\ &Z' = \mu^2 e^{2y} - \nu^2 e^{2z}. \end{aligned} \tag{4}$$

e4

Our main result is the following one.

⟨t1⟩

Theorem 1. *The orbits of the differential system (4) are one of the following three types.*

(a) *Equilibrium points: system (4) has a surface filled of equilibria.*
(b) *Periodic orbits: system (4) has a surface filled of periodic orbits with the excepcion of an equilibrium point, a center.*
(c) *All the orbits which are not in the surfaces of statements (a) and (b) are orbits which come and go to infinity.*

Theorem 1 is proved in the next section.

Of course the Bianchi VII universe have been studied by several authors. Thus, for instance Petrov [4], Terzis and Cristodoulakis [5], and others see the references cited in these two works. As far as I know in these works the authors studied particular analytical solutions of the Bianchi VII universe. That is, until now the main results on the Bianchi VII universe are quantitative results. Here we study all the qualitative solutions of the Bianchi VII universe.

2. The proof of Theorem 1

The differential system (4) has two independent first integrals

$$\begin{aligned} &H_1 = XY + XZ + YZ - \left(\mu e^y - \nu e^x\right)^2, \quad \text{and}\\ &H_2 = Y + Z. \end{aligned}$$

Recall that we want to study the dynamics of the differential system (4) satisfying $H_1 = 0$.

The differential system (4) has a surface filled with equilibria. Indeed, the points of the surface

$$(x, X, y, Y, z, Z) = \left(x, 0, z + \ln\frac{\nu}{\mu}, 0, z, 0\right),$$

are equilibrium points, as it is easy to check. This proves statement (a) of Theorem 1.

Since the variables x and X do not appear in the last four equations of the differential system (4), we restrict our attention to study the dynamics of the differential system

(5) e5
$$\begin{aligned} y' &= Y, \\ Y' &= \nu^2 e^{2z} - \mu^2 e^{2y}, \\ z' &= Z, \\ Z' &= \mu^2 e^{2y} - \nu^2 e^{2z}. \end{aligned}$$

The solutions $(y(t), Y(t), z(t), Z(t))$ of the differential system (5) satisfies that $Y(t) + Z(t) = \text{constant} = h_2$, because H_2 is a first integral. So the differential system (5) reduces on the invariant hypersurface $Y + Z = h_2$ to the differential system

(6) e6
$$\begin{aligned} y' &= Y, \\ Y' &= \nu^2 e^{2z} - \mu^2 e^{2y}, \\ z' &= h_2 - Y. \end{aligned}$$

Therefore $y' + z' = h_2$, and consequently

(7) e7
$$y(t) + z(t) = h_2 t + k,$$

where k is a constant.

From (7) it follows that if the constant $h_2 \neq 0$ then the orbits of system (4) come from the infinity and go to the infinity. This will prove statement (c) of Theorem 1 once we have completed the proof of its statement (b).

If the constant $h_2 = 0$, from (7) we have that $z(t) = k - y(t)$. Substituting $z(t)$ into the differential system (6) this system reduces to the differential system

(8) e8
$$\begin{aligned} y' &= Y, \\ Y' &= \nu^2 e^{2(k-y)} - \mu^2 e^{2y}. \end{aligned}$$

This differential system has the first integral

$$H_3 = Y^2 + \mu^2 e^{2y} + \nu^2 e^{2(k-y)},$$

because

$$\frac{dH_3}{dt} = \frac{\partial H_3}{dy} y' + \frac{\partial H_3}{dY} Y = 0.$$

The differential system (8) has a unique equilibrium point, namely

(9) e9
$$(y, Y) = \left(\frac{K}{2} - \frac{1}{4}\ln\frac{\mu^2}{\nu^2}, 0\right).$$

The eigenvalues of the Jacobian matrix of the differential system (8) at this equilibrium point are

$$\pm 2e^{K/2}\sqrt[4]{\frac{\mu^2}{\nu^2}}\, i,$$

where $i = \sqrt{-1}$ is the imaginary unit. Since the linear terms of the differential system (8) are the dominant terms near the equilibrium point (9), and the system formed only by the linear terms have a center, we have that the orbits of system (8) in a small neighborhood of the equilibrium (9) rotates around this equilibrium. So the local phase portrait at the equilibrium point (9) is either a weak focus or, a center. But since the differential system (8) has a first integral defined in the whole plane (y, Y) this equilibrium point is a center, otherwise by continuity the first integral H_3 will be constant in a neighborhood of the equilibrium point taking the value of H_3 at the equilibrium, and the first integral is not locally constant at the equilibrium.

The equilibrium point (9) of the differential system (8) having a neighborhood filled with periodic orbits surrounding the equilibrium point, is called a center. Now we shall show that all the orbits of system (8) with the exception of the equilibrium (9) are periodic surrounding this equilibrium. Indeed, the curves $H_3(y, Y) = h_3 > 0$ with the exception of $h_3 = 2e^K\sqrt{\mu^2\nu^2}$ which reduces to the equilibrium point (9) are closed or empty, and of course when they are closes are periodic orbits.

Now it remains to see that when $h_2 = 0$ for the solutions of system of the differential system (5) which are an equilibrium point and periodic orbits, the extended solutions to the differential system (4) remains of the same type, and this follows directly from the two first equations of the differential system (4). This proves statement (b) of Theorem 1. In summary Theorem 1 is proved.

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