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# RATIONAL FIRST INTEGRALS FOR POLYNOMIAL VECTOR FIELDS ON ALGEBRAIC HYPERSURFACES OF $\mathbb{R}^{n+1}$

JAUME LLIBRE* and YUDY BOLAÑOS†
*Departament de Matemàtiques,*
*Universitat Autònoma de Barcelona, 08193 Bellaterra,*
*Barcelona, Catalonia, Spain*
**jllibre@mat.uab.cat*
†*ymbolanos@gmail.com*

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Using sophisticated techniques of Algebraic Geometry, Jouanolou in 1979 showed that if the number of invariant algebraic hypersurfaces of a polynomial vector field in $\mathbb{R}^n$ of degree $m$ is at least $\binom{n+m-1}{n} + n$, then the vector field has a rational first integral. Llibre and Zhang used only Linear Algebra to provide a shorter and easier proof of the result given by Jouanolou. We use ideas of Llibre and Zhang to extend the Jouanolou result to polynomial vector fields defined on algebraic regular hypersurfaces of $\mathbb{R}^{n+1}$, this extended result completes the standard results of the Darboux theory of integrability for polynomial vector fields on regular algebraic hypersurfaces of $\mathbb{R}^{n+1}$.

*Keywords*: Darboux theory of integrability; rational first integrals; polynomial vector fields; algebraic hypersurfaces.



## 1. Introduction

In many branches of applied sciences appear nonlinear ordinary differential equations. For a differential system or a vector field defined in $\mathbb{R}^2$ the existence of a first integral determines completely its phase portrait. In $\mathbb{R}^n$ with $n > 2$ the existence of a first integral of a vector field reduces the study of its dynamics in one dimension, with the time real or complex, respectively. So the following natural question arises: Given a vector field on $\mathbb{R}^n$, how to recognize if this vector field has a first integral? This question has no satisfactory answers up to now. Many different methods have been used for studying the existence of first integrals of vector fields. Some of these methods are based on: Noether symmetries [Cantrijn & Sarlet, 1981], the Darboux theory of integrability [Darboux, 1878], the Lie symmetries [Olver, 1986], the Painlevé analysis [Bountis *et al.*, 1984], the use of Lax pairs [Lax, 1968], the direct method [Giacomini *et al.*, 1991; Hietarinta, 1987] and, the linear compatibility analysis method [Strelcyn & Wojciechowski, 1988], the Carleman embedding procedure [Andrade & Rauth, 1981; Carleman, 1932], the quasimonomial formalism [Bountis *et al.*, 1984], etc. For polynomial vector fields and rational first integrals the best answer to this question was given by Jouanolou [1979] in 1979 inside the Darboux theory of integrability. This theory of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have.

The objective of this paper is to extend the result of Jouanolou on the existence of rational first integrals of polynomial vector fields of $\mathbb{R}^{n+1}$ to polynomial vector fields defined on regular algebraic hypersurfaces of $\mathbb{R}^{n+1}$.