DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **17**, Number **3**, March **2007**

Website: http://aimSciences.org

pp. **529–540**

# 3–DIMENSIONAL HOPF BIFURCATION VIA AVERAGING THEORY

Jaume Llibre
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain.

Claudio A. Buzzi and Paulo R. da Silva
Departamento de Matemática
Universidade Estadual Paulista–UNESP
15054-000 São José do Rio Preto, São Paulo, Brazil.

Abstract. We consider the Lorenz system $\dot{x} = \sigma(y - x)$, $\dot{y} = rx - y - xz$ and $\dot{z} = -bz + xy$; and the Rössler system $\dot{x} = -(y + z)$, $\dot{y} = x + ay$ and $\dot{z} = b - cz + xz$. Here, we study the Hopf bifurcation which takes place at $q_\pm = (\pm\sqrt{br - b}, \pm\sqrt{br - b}, r - 1)$, in the Lorenz case, and at $s_\pm = \left(\frac{c+\sqrt{c^2-4ab}}{2}, -\frac{c+\sqrt{c^2-4ab}}{2a}, \frac{c\pm\sqrt{c^2-4ab}}{2a}\right)$ in the Rössler case. As usual this Hopf bifurcation is in the sense that an one–parameter family in $\varepsilon$ of limit cycles bifurcates from the singular point when $\varepsilon = 0$. Moreover, we can determine the kind of stability of these limit cycles. In fact, for both systems we can prove that all the bifurcated limit cycles in a neighborhood of the singular point are either a local attractor, or a local repeller, or they have two invariant manifolds, one stable and the other unstable, which locally are formed by two 2–dimensional cylinders. These results are proved using averaging theory. The method of studying the Hopf bifurcation using the averaging theory is relatively general and can be applied to other 3– or $n$–dimensional differential systems.

1. **Introduction.** The main goal of this work is to study the Hopf bifurcation occurring in vector fields in $\mathbb{R}^3$ via averaging theory. We will investigate the quadratic systems in $\mathbb{R}^3$ with a singular point at the origin $(0, 0, 0)$ having linear part with eigenvalues $\varepsilon a \pm c\,i$ and $\varepsilon d$, where $\varepsilon$ is a small parameter.

With an appropriate change of coordinates we can assume that our system has its linear part in the real Jordan normal form, that is

$$\begin{aligned}
\dot{U} &= \varepsilon aU - cV + \sum_{i+j+k=2} A_{ijk}U^iV^jW^k,\\
\dot{V} &= cU + \varepsilon aV + \sum_{i+j+k=2} B_{ijk}U^iV^jW^k,\\
\dot{W} &= \varepsilon dW + \sum_{i+j+k=2} C_{ijk}U^iV^jW^k.
\end{aligned} \tag{1}$$

2000 *Mathematics Subject Classification.* 37G15, 37D45.
*Key words and phrases.* Hopf bifurcation, Lorenz system, averaging theory.
* The first and the third authors are partially supported by PROCAD–CAPES–Brazil 0092/01-0, tipo I, and by a FAPESP–BRAZIL grant 10246–2. The second author is partially supported by MCYT/FEDER grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550. All authors are supported by the joint project CAPES–MECD grant 071/04 and HBP2003–0017.