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Analytical study of a triple Hopf bifurcation in a tritrophic food chain model

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ABSTRACT

We provide an analytical proof of the existence of a stable periodic orbit contained in the region of coexistence of the three species of a tritrophic chain. The method used consists in analyzing a triple Hopf bifurcation. For some values of the parameters three limit cycles born via this bifurcation. One is contained in the plane where the top-predator is absent. Another one is not contained in the domain of interest where all variables are positive. The third one is contained where the three species coexist. The techniques for proving these results have been introduced in previous articles by the second author and are based on the averaging theory of second-order. Existence of this triple Hopf bifurcation has been previously discovered numerically by Kooij et al.

1. Introduction

During these last 80 years, after the seminal works of Lotka [17] and Volterra [27], one of the main topics in mathematical ecology has been the study of (di) trophic food chains. This has been made by analyzing many different planar differential systems under the common name of prey–predator models, for instance see [1,7]. The existence of limit cycles, attractors, and several kind of bifurcations are the characteristics of those models which have been used to explain the complex behaviors observed in such systems.

In the late seventies some interest in the mathematics of tritrophic food chain models (composed of prey, predator, and top-predator) appeared, see for example [9,10,8] and Predator–Prey models with parasitic infection [11]. The model we analyze in this article describes a tritrophic food chain composed of a logistic prey (x), a Holling type II predator (y), and a Holling type II top-predator (z). After a rescaling of the variables, it is given by the following system of ordinary differential equations (see [12,22,16,13,20] for more details):

$$
\begin{aligned}
e^2\dot{x} &= x\left(\rho - \frac{x}{k} - \frac{a_1 y}{b_1 + x}\right),\\
e\dot{y} &= y\left(\frac{a_1 x}{b_1 + x} - \frac{a_2 z}{b_2 + y} - d_1\right),\\
\dot{z} &= z\left(\frac{a_2 y}{b_2 + y} - d_2\right).
\end{aligned}
\tag{1}
$$

In order to preserve the biological meaning of the model, the nine parameters of this system are assumed to be strictly positive. Similar types of systems have been studied in the case when time scales of the variables are different so that methods of approximations of slow–fast systems can be applied [25,26]. We emphasize that we do not need here this type of approximation. To clarify possible applications of our result, we focus in a range where the population of the superpredator z remains small (near

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