LIMIT CYCLES BIFURCATING FROM THE PERIODIC ANNULUS OF CUBIC HOMOGENEOUS POLYNOMIAL CENTERS

JAUME LLIBRE [1], BRUNO D. LOPES [1,2] AND JAIME R. DE MORAES [2]

Abstract. We obtain an explicit polynomial whose simple positive real roots provide the limit cycles which bifurcate from the periodic orbits of any cubic homogeneous polynomial center when it is perturbed inside the class of all polynomial differential systems of degree n.

1. Introduction and Statement of the Main Results

One of the main goals in the qualitative theory of real planar differential systems is the determination of their limit cycles. It is well known that perturbing the periodic orbits of a center often produces limit cycles, see for instance [1, 2, 12]. One of the first in studying these perturbations was Pontrjagin [10]. These last years this problem has been studied by many authors see the second part of the book [5] and the hundreds of references quoted there.

Hilbert in 1900 was interested in the maximum number of the limit cycles that a polynomial differential system of a given degree can have. This problem is the well-known 16-th Hilbert problem, which together with the Riemann conjecture are the two problems of the famous list of 23 problems of Hilbert which remain open. See for more details [7] and [13].

There exist several methods to study the number of limit cycles that bifurcate from the periodic annulus of a center, such as the *Poincaré return map*, the *Poincaré–Melnikov integrals*, the *Abelian integrals*, the *inverse integrating factor*, and the *averaging theory*. In the plane all of them are essentially equivalent.

There are few works trying to study this problem for homogeneous cubic polynomial differential systems. Our main objetive will be to solve this problem for the cubic homogeneous polynomial differential systems.

In [6] the authors classified all the cubic homogeneous polynomial differential systems. In [8] the authors proved that any real planar cubic homogeneous polynomial differential system having a center can be written

2010 *Mathematics Subject Classification.* 37C10, 34H99.

Key words and phrases. polinomial vector field, limit cycle, averaging method, periodic orbit, isochronous center, homogeneous cubic centers.

as

$$
\begin{array}{lcccl}
\dot{x} & = & ax^3 + (b-3\alpha\mu)x^2y - axy^2 - \alpha y^3 & = & P(x,y), \\
\dot{y} & = & \alpha x^3 + ax^2y + (b+3\alpha\mu)xy^2 - ay^3 & = & Q(x,y),
\end{array} \tag{1}
$$

with $\alpha \in \{-1, 1\}$, $a, b, \mu \in \mathbb{R}$ and $\mu > -1/3$, after doing an affine change of variables and a rescaling of the time.

It is known that the maximum number of limit cycles which bifurcate from the periodic orbits of a cubic homogeneous center (1) using perturbations of first order inside the class of all polynomial differential systems of degree n is $[(n-1)/2]$, see for details statement (c) of Theorem A of [9]. Here $[x]$ denotes the integer part function of x.

The objetive of this work is *to provide an explicit polynomial whose real positive simple zeros gives the exact number of limit cycles which bifurcate, at first order in the perturbation parameter, from the periodic orbits of any cubic homogeneous center* (1).

More precisely consider the system.

$$
\begin{array}{lcl}
\dot{x} & = & ax^3 + (b-3\alpha\mu)x^2y - axy^2 - \alpha y^3 + \varepsilon p(x,y), \\
\dot{y} & = & \alpha x^3 + ax^2y + (b+3\alpha\mu)xy^2 - ay^3 + \varepsilon q(x,y),
\end{array} \tag{2}
$$

where

$$
p(x,y) = \sum_{i=0}^{n} p_i(x,y), \quad q(x,y) = \sum_{i=0}^{n} q_i(x,y), \tag{3}
$$

p_i, q_i are homogeneous polynomials of degree i, and ε is a small parameter.

Define the following functions

$$
\begin{array}{rcl}
f_1(\theta) & = & -a\sin^4\theta + a\cos^4\theta + (b-3\alpha\mu+\alpha)\sin\theta\cos^3\theta \\
 & & +(b+3\alpha\mu-\alpha)\sin^3\theta\cos\theta, \\
g_1(\theta) & = & \alpha\left(6\mu\sin^2\theta\cos^2\theta + \sin^4\theta + \cos^4\theta\right), \\
k(\theta) & = & \exp\left(\displaystyle\int_0^\theta \frac{f_1(s)}{g_1(s)}ds\right), \\
B_i(\theta) & = & Q(\cos\theta,\sin\theta)p_i(\cos\theta,\sin\theta) - P(\cos\theta,\sin\theta)q_i(\cos\theta,\sin\theta).
\end{array}
$$

In sequel we state our main result where the function $M(\theta)$ is defined in (9), we do not provide it here due to its length.

Theorem 1. *For $|\varepsilon| > 0$ sufficiently small and for every positive simple zero r_0^* of the polynomial*

$$
\mathcal{F}(r_0) = \frac{1}{2\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} r_0^{2k-1}\int_0^{2\pi} A_{2k+1}(\theta)d\theta,
$$

where

$$
A_i(\theta) = \frac{B_i(\theta)k(\theta)^{i-2}}{g_1(\theta)^2 M(\theta)},
$$

for $i = 1, 2, ..., \left[\frac{n-1}{2}\right]$, *the perturbed systems* (2) *has a limit cycle bifurcating from the periodic orbit* $r(\theta, r_0^*) = k(\theta)r_0^*$ *of the period annulus of the center* (1) *using the averaging theory of first order. In particular the perturbed systems* (2) *has at most* $\left[\frac{n-1}{2}\right]$ *limit cycles.*

Theorem 1 is proved in Section 3. In Section 4 we provide an example that illustrates Theorem 1 with $n = 5$. We obtain two limit cycles.

2. Preliminaries

In this section we give some known results that we shall need for proving Theorem 1.

Consider a system in the form

$$\dot{\mathrm{x}} = F_0(t, \mathrm{x}) + \varepsilon F_1(t, \mathrm{x}) + \mathcal{O}(\varepsilon^2), \tag{4}$$

where $\varepsilon \neq 0$ is sufficiently small and the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, $T-$periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. We assume that the unperturbed system

$$\dot{\mathrm{x}} = F_0(t, \mathrm{x}) \tag{5}$$

has a submanifold of periodic solutions of dimension n.

Consider $x(t, z, \varepsilon)$ the solution of system (5) such that $\mathrm{x}(0, \mathrm{z}, \varepsilon) = z$. The linearization of the unperturbed system along a periodic solution $\mathrm{x}(t, \mathrm{z}, 0)$ is given by

$$\dot{\mathrm{y}} = D_{\mathrm{x}}F_0(t, \mathrm{x}(t, \mathrm{x}, 0))\mathrm{y}. \tag{6}$$

In sequel we denote by $M_{\mathrm{z}}(t)$ the fundamental matrix of the linearized system (6) such that $M_{\mathrm{z}}(0)$ is the identity.

We suppose that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $z \in \mathrm{Cl}(V)$, $\mathrm{x}(t, \mathrm{z}, 0)$ is $T-$periodic, where $\mathrm{x}(t, z, 0)$ denotes the solution of the unperturbed system (5). Here $\mathrm{Cl}(V)$ denotes the closure of V. We have that the set $\mathrm{Cl}(V)$ is *isochronous* for system (5), i.e. it is formed only by periodic orbits with period T.

The next result is the averaging theorem for studying the bifurcation of $T-$periodic solutions of system (4) from the periodic solutions $\mathrm{x}(t, \mathrm{z}, 0)$ contained in $\mathrm{Cl}(V)$ of system (5) when $|\varepsilon| > 0$ is sufficiently small. See [3] for a proof. For more details on the averaging theory see [4] and the book [11].

Theorem 2 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set* V *with* $\mathrm{Cl}(V) \subset \Omega$ *such that for each* $\mathrm{z} \in \mathrm{Cl}(V)$, *the solution* $\mathrm{x}(r, \mathrm{z}, 0)$ *is* $T-$*periodic. Consider the function* $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$

$$\mathcal{F}(\mathrm{z}) = \frac{1}{T}\int_0^T M_{\mathrm{z}}^{-1}(t)F_1(t, \mathrm{x}(t, \mathrm{z}, 0))dt. \tag{7}$$

Then the following statements hold.

(i) *If there exists* $\mathbf{a} \in V$ *with* $\mathcal{F}(\mathbf{a}) = 0$ *and* $\det((\partial \mathcal{F}/\partial \mathrm{z})(\mathbf{a})) \neq 0$ *then there exists a* $T-$*periodic solution* $\mathrm{x}(t, \varepsilon)$ *of system* (4) *such that* $\mathrm{x}(0, \varepsilon) \to \mathbf{a}$ *when* $\varepsilon \to 0$.

(ii) *The kind of the stability of the periodic solution* $\mathrm{x}(t, \varepsilon)$ *is given by the eigenvalues of the Jacobian matrix* $((\partial \mathcal{F}/\partial \mathrm{z})(\mathbf{a}))$.

3. Proof of Theorem 1

The next result follows easily.

Lemma 3. *Let* $P_k(x, y)$ *and* $P_3(x, y)$ *be homogeneous polynomials of degree* k *and* 3 *respectively, where* $(x, y) \in \mathbb{R}^2$. *Thus in polar coordinates* $x = r\cos\theta$ *and* $y = r\sin\theta$ *we have*

$$\begin{aligned} P_k(r\cos\theta, r\sin\theta)P_3(r\cos\theta, r\sin\theta) &= (-1)^{k+1}P_k(r\cos(\theta+\pi), r\sin(\theta+\pi)) \\ &\quad \cdot P_3(r\cos(\theta+\pi), r\sin(\theta+\pi)). \end{aligned}$$

Now we pass system (2) to polar coordinates taking $x = r\cos\theta$, $y = r\sin\theta$ and we obtain

$$\begin{aligned} \dot{r} &= r^3 f_1(\theta) + \varepsilon(\cos\theta\ p(r\cos\theta, r\sin\theta) + \sin\theta\ q(r\cos\theta, r\sin\theta)), \\ \dot{\theta} &= r^2 g_1(\theta) + \varepsilon\frac{1}{r}(\cos\theta\ q(r\cos\theta, r\sin\theta) - \sin\theta\ p(r\cos\theta, r\sin\theta)). \end{aligned}$$

Note that $g_1(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$. Thus we take the quotient $\dot{r}/\dot{\theta}$ and we get the differential equation

$$\frac{dr}{d\theta} = F_0(r, \theta) + \varepsilon F_1(r, \theta) + \mathcal{O}(\varepsilon^2), \tag{8}$$

in the standard form for applying the averaging theory of first order, where

$$\begin{aligned} F_0(r, \theta) &= \frac{f_1(\theta)}{g_1(\theta)}r, \\ F_1(r, \theta) &= \frac{1}{r^5 g_1(\theta)^2}(Q(r\cos\theta, r\sin\theta)p(r\cos\theta, r\sin\theta) \\ &\quad -P(r\cos\theta, r\sin\theta)q(r\cos\theta, r\sin\theta)). \end{aligned}$$

Note that the differential equation (8) satisfies the assumptions of Theorem 2. Consider $r(\theta, r_0)$ the periodic solution of the differential equation $\dot{r} = rf_1(\theta)/g_1(\theta)$ such that $r(0, r_0) = r_0$. Solving this differential equation we obtain

$$r(\theta, r_0) = k(\theta)r_0 = r_0 e^{k_1(\theta)}k_2(\theta),$$

where

$$k_1(\theta) = -\frac{a\alpha}{2R}\left(\frac{(-3\mu + R - 1)\tan^{-1}\left(\dfrac{\tan\theta}{\sqrt{3\mu - R}}\right)}{\sqrt{3\mu - R}}\right.$$

$$
\left.+\frac{(3\mu+R+1)\tan^{-1}\left(\dfrac{\tan\theta}{\sqrt{3\mu+R}}\right)}{\sqrt{3\mu+R}}\right),
$$

$$
\begin{aligned}
k_2(\theta) &= \sqrt{\sec^2\theta}(3\mu-R)^{\frac{R-\alpha b}{4R}}(3\mu+R)^{\frac{\alpha b+R}{4R}}\left(\tan^2\theta+3\mu-\alpha R\right)^{\frac{1}{4}\left(\frac{b}{R}-1\right)} \\
&\quad \cdot\left(\tan^2\theta+3\mu+\alpha R\right)^{-\frac{b+R}{4R}}, \\
R &= \sqrt{9\mu^2-1}.
\end{aligned}
$$

Solving the variational equation (6) for our differential equation (8) we get that the fundamental matrix of (6) is

$$
\begin{aligned}
M(\theta) &= \Bigg(-2a\alpha(-3\mu+R-1)\sqrt{3\mu+R}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{3\mu-R}}\right) \\
&\quad -2a\alpha\sqrt{3\mu-R}(3\mu+R+1)\tan^{-1}\left(\frac{\tan\theta}{\sqrt{3\mu+R}}\right)+\alpha\sqrt{3\mu-R} \\
&\quad \cdot\sqrt{3\mu+R}(b-\alpha R)\log\left(\tan^2\theta+3\mu-R\right)-\alpha\sqrt{3\mu-R} \\
&\quad \cdot\sqrt{3\mu+R}(b+\alpha R)\log\left(\tan^2\theta+3\mu+R\right)+\sqrt{3\mu-R}\sqrt{3\mu+R} \\
&\quad \cdot(R-\alpha b)\log(3\mu-R)+\sqrt{3\mu-R}\sqrt{3\mu+R}(\alpha b+R)\log(3\mu+R) \\
&\quad +2R\sqrt{3\mu-R}\sqrt{3\mu+R}\log\left(\sec^2\theta\right) \\
&\quad +4R\sqrt{3\mu-R}\sqrt{3\mu+R}\Bigg)\Big/\left(4R\sqrt{3\mu-R}\sqrt{3\mu+R}\right).
\end{aligned}
\tag{9}
$$

Note that $M(\theta)$ does not depend on r_0. Using the polynomials p and q given in (3) and system (1) we have that the integrant of the integral (7) for our differential equation is

$$
M^{-1}(\theta)F_1(\theta,r(\theta,r_0))=\frac{F_1(\theta,r(\theta,r_0))}{M(\theta)}=\frac{h(r,\theta)}{r^5g_1(\theta)^2M(\theta)}, \tag{10}
$$

where

$$
h(r,\theta)=Q(r\cos\theta,r\sin\theta)p(r\cos\theta,r\sin\theta)-P(r\cos\theta,r\sin\theta)q(r\cos\theta,r\sin\theta).
$$

Since p and q are sum of homogeneous polynomials (see eq. (3)), we can rewrite the equality (10) as follows

$$
\begin{aligned}
M^{-1}(\theta)F_1(\theta,r(\theta,r_0)) &= \sum_{i=0}^{n}\frac{B_i(\theta)}{g_1(\theta)^2M(\theta)}r(\theta,r_0)^{i-2} \\
&= \sum_{i=0}^{n}r_0^{i-2}\frac{B_i(\theta)(e^{k_1(\theta)}k_2(\theta))^{i-2}}{g_1(\theta)^2M(\theta)} \\
&= \sum_{i=0}^{n}r_0^{i-2}A_i(\theta).
\end{aligned}
$$

Computing the integral (7) we have that

$$\mathcal{F}(r_0)=\frac{1}{2\pi}\int_0^{2\pi} M^{-1}(\theta)F_1(\theta,r(\theta,r_0)d\theta=\frac{1}{2\pi}\sum_{i=0}^{n} r_0^{i-2}\int_0^{2\pi} A_i(\theta)d\theta.$$

If i is even then Lemma 3 implies $B_i(\theta)=-B_i(\theta+\pi)$. Thus since $k_1(\theta)=k_1(\theta+\pi)$, $k_2(\theta)=k_2(\theta+\pi)$ and $M(\theta)=M(\theta+\pi)$ we easily obtain

$$\begin{aligned}
\int_\pi^{\frac{3\pi}{2}} A_i(\theta)d\theta &= \int_\pi^{\frac{3\pi}{2}} \frac{B_i(\theta)(e^{k_1(\theta)}k_2(\theta))^{i-2}}{g_1(\theta)^2M(\theta)}d\theta\\
&= \int_0^{\frac{\pi}{2}} \frac{B_i(\theta+\pi)(e^{k_1(\theta+\pi)}k_2(\theta+\pi))^{i-2}}{g_1(\theta+\pi))^2M(\theta+\pi))}d\theta\\
&= \int_0^{\frac{\pi}{2}} -\frac{B_i(\theta)(e^{k_1(\theta)}k_2(\theta))^{i-2}}{g_1(\theta)^2M(\theta)}d\theta\\
&= -\int_0^{\frac{\pi}{2}} A_i(\theta)d\theta,\\
\int_{\frac{3\pi}{2}}^{2\pi} A_i(\theta)d\theta &= \int_{\frac{3\pi}{2}}^{2\pi} \frac{B_i(\theta)(e^{k_1(\theta)}k_2(\theta))^{i-2}}{g_1(\theta)^2M(\theta)}d\theta\\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{B_i(\theta+\pi)(e^{k_1(\theta+\pi)}k_2(\theta+\pi))^{i-2}}{g_1(\theta+\pi))^2M(\theta+\pi))}d\theta\\
&= \int_{\frac{\pi}{2}}^{\pi} -\frac{B_i(\theta)(e^{k_1(\theta)}k_2(\theta))^{i-2}}{g_1(\theta)^2M(\theta)}d\theta\\
&= -\int_{\frac{\pi}{2}}^{\pi} A_i(\theta)d\theta.
\end{aligned}$$

Therefore for i even we have

$$\int_0^{2\pi} A_i(\theta)d\theta=0.$$

Analogously if i is odd then Lemma 3 implies $B_i(\theta)=B_i(\theta+\pi)$. Thus we easily can check that

$$\int_0^{\frac{\pi}{2}} A_i(\theta)d\theta=\int_\pi^{\frac{3\pi}{2}} A_i(\theta)d\theta \quad\text{and}\quad \int_{\frac{\pi}{2}}^{\pi} A_i(\theta)d\theta=\int_{\frac{3\pi}{2}}^{2\pi} A_i(\theta)d\theta.$$

So we have

$$\int_0^{2\pi} A_i(\theta)d\theta=2\int_0^{\pi} A_i(\theta)d\theta\neq 0,$$

because A_i is a $\pi-$periodic even function. So the function $\mathcal{F}$ can be written in the following way

$$\mathcal{F}(r_0)=\frac{1}{2\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} r_0^{2k-1}\int_0^{2\pi} A_{2k+1}(\theta)d\theta. \tag{11}$$

Note that the coefficients $A_{2k+1}(\theta)$ in (11) are linearly independent because the polynomials p_i and q_i are linearly independents. Thus the averaged function $\mathcal{F}$ has at most $[(n-1)/2]$ simple zeros which correspond to the limit cycles of system (2) and Theorem 1 is proved.

4. Example

In this section we present an example that illustrates Theorem 1.

Consider the cubic polynomial homogeneous center

$$\dot{x} = -y^3, \quad \dot{y} = x^3,$$

and its perturbation

$$\dot{x} = -y^3 + \varepsilon(a_1 x + a_2 x^3 + a_3 x^5), \quad \dot{y} = x^3. \tag{12}$$

Passing system (12) to the polar coordinates we get

$$\begin{aligned} \dot{r} &= r^3(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta) + \varepsilon r\cos^2\theta\,(a_1 + a_2 r^2\cos^2\theta + a_3 r^4\cos^4\theta), \\ \dot{\theta} &= r^2(\sin^4\theta + \cos^4\theta) - \varepsilon\sin\theta\cos\theta\,(a_1 + a_2 r^2\cos^2\theta + a_3 r^4\cos^4\theta). \end{aligned}$$

Taking the quotient $\dot{r}/\dot{\theta}$ we obtain the following system in the standard form of Theorem 2 for applying the averaging theory

$$\frac{dr}{d\theta} = F_0(r,\theta) + \varepsilon F_1(r,\theta) + \mathcal{O}(\varepsilon^2), \tag{13}$$

where

$$\begin{aligned} F_0(r,\theta) &= \frac{r\sin\theta\cos^3\theta - r\sin^3\theta\cos\theta}{\sin^4\theta + \cos^4\theta}, \\ F_1(r,\theta) &= \frac{\cos^4\theta(a_1 + a_2 r^2\cos^2\theta + a_3 r^4\cos^4\theta)}{r(\sin^4\theta + \cos^4\theta)^2}. \end{aligned}$$

Thus for system (13) we have

$$\begin{aligned} k(\theta) &= \frac{\sqrt{2}}{\sqrt[4]{\cos(4\theta) + 3}}, \\ M(\theta) &= \frac{1}{4}(-\log(\cos(4\theta) + 3) + 4 + \log 4), \end{aligned}$$

and the integrant of the integral (7) of system (13) is

$$\frac{A(\theta) + B(\theta)r_0^2 + C(\theta)r_0^4}{r_0},$$

where

$$\begin{aligned} A(\theta) &= a_1\frac{32\sqrt{2}\cos^4\theta}{(\cos(4\theta)+3)^{7/4}(-\log(\cos(4\theta)+3) + 4 + \log 4)}, \\ B(\theta) &= a_2\frac{64\sqrt{2}\cos^6\theta}{(\cos(4\theta)+3)^{9/4}(-\log(\cos(4\theta)+3) + 4 + \log 4)}, \\ C(\theta) &= a_3\frac{128\sqrt{2}\cos^8\theta}{(\cos(4\theta)+3)^{11/4}(-\log(\cos(4\theta)+3) + 4 + \log 4)}. \end{aligned}$$

Computing numerically the integral (7) for system (13) we obtain

$$\mathcal{F}(r_0) = \frac{3.72731...\, a_1 + 3.34745...\, a_2\, r_0^2 + 3.10284...\, a_3\, r_0^4}{r_0}.$$

Taking

$$a_1 = \frac{-1}{3.72731...}, \quad a_2 = \frac{2}{3.34745...} \quad \text{and} \quad a_3 = \frac{-0.1}{3.10284...},$$

it is easy to check that the function $\mathcal{F}$ has two positive simple zeros given by

$$r_0^* = 0.716357... \quad \text{and} \quad r_0^{**} = 4.41439...$$

which correspond to two limit cycles of the perturbed system (12) with $\varepsilon \neq 0$ sufficiently small.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant number MTM2009-03437, by an AGAUR grant number 2014SGR-568, by an ICREA Academia, two FP7+PEOPLE+2012+IRSES numbers 316338 and 318999, and FEDER-UNAB10-4E-378. The second author is supported by CAPES/GDU - 7500/13-0.

References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer, *Theory of bifurcations of dynamic systems on a plane*, Halsted Press [A division of John Wiley & Sons], New York-Toronto, Ont., 1973, Translated from the Russian.

[2] T.R. Blows and L.M. Perko, *Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems*, SIAM Rev. **36** (1994), 341–376.

[3] A. Buică, J. P. Françoise, and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Commun. Pure Appl. Anal. **6** (2007), 103–111.

[4] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[5] C. Christopher and C. Li, *Limit cycles in differential equations*, Birkhauser, Boston, 1999.

[6] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, J. Math. Anal. Appl. **147** (1990), 420–448.

[7] C. Hilbert, *Mathematische probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ttingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.

[8] C. Li and J. Llibre, *Cubic homogeneous polynomial centers*, Publ. Mat. **58** (2014), 297–308.

[9] W. Li, J. Llibre, J. Yang, and Z. Zhang, *Limit cycles bifurcating from the period annulus of quasi–homogeneous centers*, J. Dynam. Differential Equations **21** (2009), 133–152.

[10] L. S. Pontrjagin, *Über autoschwingungssysteme, die den hamiltonshen nahe liegen*, Physikalische Zeitschrift der Sowjetunion **6** (1934), 25–28.

[11] J.A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, vol. 59, Springer, New York, 2007.

[12] Y. Shao and K. Wu, *Bifurcation of limit cycles for cubic reversible systems*, Electron. J. Differential Equations **2014** (2014), No. 96, 10 pp.
[13] S. Smale, *Mathematical problems for next century*, Math. Intelligencer **20** (1998), 7–15.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.
E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matemática – IBILCE–UNESP, Rua C. Colombo, 2265, CEP 15054–000 S. J. Rio Preto, São Paulo, Brazil.
E-mail address: jrezmoraes@gmail.com
E-mail address: blopes@mat.uab.cat