# LIMIT CYCLES BIFURCATING FROM THE PERIODIC ORBITS OF THE WEIGHT-HOMOGENEOUS POLYNOMIAL CENTERS OF WEIGHT-DEGREE 3 

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#### Abstract

In this paper we obtain two explicit polynomials, whose simple positive real roots provide the limit cycles which bifurcate from the periodic orbits of a family of polynomial differential centers of order 5 , when this family is perturbed inside the class of all polynomial differential systems of order 5 , whose average function of first order is not zero. Then the maximum number of limit cycles that bifurcate from these periodic orbits is 6 and it is reached.

The family of centers studied completes the study about the limit cycles which can bifurcate from the periodic orbits of all centers of the weight-homogeneous polynomial differential systems of weight-degree 3 , when we perturb them inside the class of all polynomial differential systems having the same degree, and whose average function of first order is not zero.


## 1. Introduction and Statement of the Main Results

One of the main goals in the qualitative theory of real planar polynomial differential systems is the determination of their limit cycles. The problem to study the number of limit cycles of a polynomial differential system is strongly motivated by the 16 -th Hilbert's problem (1900). For more details, see [8] and [13].

Many authors have studied the number of limit cycles which can bifurcate from the periodic orbits of a center of a polynomial differential system when it is perturbed up to first order in the parameter of the perturbation. This problem is well known as the weak Hilbert's problem. Look for example the references $[1,3]$.

There exist many tools for studying the maximum number of limit cycles that bifurcate from the periodic annulus of a center. Among them we have the Poincaré return map, the Poincaré-Melnikov integrals, the Abelian integrals, and the averaging theory. The last three methods are equivalent at first order, see for instance [7]. Many authors studied the weak Hilbert's

[^0]problem, see for example the second part of the book [6] and the hundreds of references quoted there.

Here we consider polynomial differential systems given by

$$
\begin{align*}
& \dot{x}=P(x, y), \\
& \dot{y}=Q(x, y), \tag{1}
\end{align*}
$$

where $P$ and $Q$ are polynomials with real coefficients. The degree of the system is the maximum of the degrees of the polynomials $P$ and $Q$.

We say that system (1) is weight-homogeneous if there exist $\left(s_{1}, s_{2}\right) \in \mathbb{N}^{2}$ and $d \in \mathbb{N}$ such that for any $\lambda \in \mathbb{R}^{+}=\{\lambda \in \mathbb{R}: \lambda>0\}$ we have

$$
P\left(\lambda^{s_{1}} x, \lambda^{s_{2}} y\right)=\lambda^{s_{1}-1+d} P(x, y), \quad Q\left(\lambda^{s_{1}} x, \lambda^{s_{2}} y\right)=\lambda^{s_{2}-1+d} Q(x, y) .
$$

The vector $\left(s_{1}, s_{2}\right)$ is called the weight-exponent of system (1) and $d$ is called weight-degree with respect to the weight-exponent $\left(s_{1}, s_{2}\right)$.

There are few works trying to study the weak Hilbert's problem for weight-homogeneous polynomial differential systems. Our main goal is to solve the weak Hilbert's problem for the weight-homogeneous polynomial differential systems of weight-degree 3 .

The classification of all centers of a planar weight-homogeneous polynomial differential systems has been made in [11] up to weight-degree 4. Particularly they provide two families of weight-homogeneous polynomial differential systems having centers with weight-degree 3 . The first family can be written as

$$
\begin{align*}
& \dot{x}=a x^{3}+(b-3 \alpha \mu) x^{2} y-a x y^{2}-\alpha y^{3}, \\
& \dot{y}=\alpha x^{3}+a x^{2} y+(b+3 \alpha \mu) x y^{2}-a y^{3}, \tag{2}
\end{align*}
$$

with $\alpha \in\{-1,1\}, a, b, \mu \in \mathbb{R}$ and $\mu>-1 / 3$, after doing an affine change of variables and a rescaling of the time. The weight-exponent of this family is $\left(s_{1}, s_{2}\right)=(1,1)$.

The second family is given by

$$
\begin{align*}
& \dot{x}=a x^{3}+b y=P(x, y),  \tag{3}\\
& \dot{y}=c x^{5}+d x^{2} y=Q(x, y),
\end{align*}
$$

with $b c \neq 0,3 a+d=0$ and $12(b c-a d)<0$. The weight-exponent of this family is $\left(s_{1}, s_{2}\right)=(1,3)$.

In [10] we provide a polynomial whose real positive simple zeros give exactly the number of limit cycles that bifurcate from the periodic orbits of system (2) when it is perturbed inside the class of all polynomial differential systems of degree $n$, and the first order average function is non-zero. In particular the maximum number of limit cycles obtained is exactly [ $n-$ $1) / 2]$, where $[x]$ denotes the integer part function of $x$.

In this paper we give two explicit polynomials whose real positive simple zeros provide the number of limit cycles which bifurcate from the periodic
orbits of the center of the weight-homogeneous polynomial differential system (3), when the first order average function is non-zero. The maximum number of limit cycles is reached when the parameter $d \neq 0$.

More precisely here we consider the polynomial differential system

$$
\begin{align*}
\dot{x} & =-d f r a c d 3 x^{3}+b y+\varepsilon p(x, y)  \tag{4}\\
\dot{y} & =c x^{5}+d x^{2} y+\varepsilon q(x, y)
\end{align*}
$$

where

$$
\begin{gather*}
p(x, y)=\sum_{k=0}^{5} p_{k}(x, y), \quad q(x, y)=\sum_{i=0}^{5} q_{k}(x, y)  \tag{5}\\
p_{k}(x, y)=\sum_{i=0}^{k} c_{k-i i} x^{k-i} y^{i}, \quad q_{k}(x, y)=\sum_{i=0}^{k} d_{k-i i} x^{k-i} y^{i},
\end{gather*}
$$

and $\varepsilon$ is a small parameter.
In [9] the authors found an upper bound for the maximum number of limit cycles of system (4). Looking at statement (c) of Theorem A of [9] with $n=5, p=3$ and $q=1$ the upper bound for the number of limit cycles of system (4) with $\varepsilon$ sufficiently small coming from the periodic orbits of the center (3) is 8 . Here we prove that the maximum number of limit cycles that system (4) can have is exactly 6 and it is reached. See Corollary 3.

In what follows we state our main results where the functions $f_{i}(\theta)$, for $i=0,1, \ldots, 9, W(\theta), g_{1}(\theta)$ and $k(\theta)$ are given in Section 3 and the functions $f_{i}^{*}(\theta)$, for $i=0,1, \ldots, 9, W^{*}(\theta), g_{1}^{*}(\theta)$ and $k^{*}(\theta)$ are given in Section 4. We do not provide these functions here due to their length.

Theorem 1. Suppose that $d \neq 0$. Let $r_{0, s}$ be a positive simple root of the polynomial

$$
\begin{equation*}
r_{0} \mathcal{F}\left(r_{0}\right)=\frac{1}{2 \pi} \sum_{k=1}^{7} r_{0}^{2 k-3} \int_{0}^{2 \pi} A_{2 k+1}(\theta) d \theta \tag{6}
\end{equation*}
$$

where

$$
A_{i}(\theta)=\frac{W(\theta) f_{i}(\theta) k(\theta)^{i-5}}{g_{1}(\theta)^{2}}
$$

Then for $|\varepsilon|>0$ sufficiently small parameter the perturbed systems (4) have a limit cycle bifurcating from the periodic orbit $r\left(\theta, r_{0, s}\right)=k(\theta) r_{0, s}$ of the period annulus of the center (3) if the first order average function is nonzero. In particular, the polynomial (6) has at most 6 positive simple real roots and they are reached.

Theorem 2. Suppose that $d=0$. Let $r_{0, s}$ be a positive simple root of the polynomial

$$
\begin{equation*}
r_{0} \mathcal{F}^{*}\left(r_{0}\right)=\frac{1}{2 \pi}\left(\sum_{k=1}^{5} r_{0}^{2 k-3} \int_{0}^{2 \pi} A_{2 k+1}^{*}(\theta) d \theta+r_{0}^{11} \int_{0}^{2 \pi} A_{15}^{*}(\theta) d \theta\right) \tag{7}
\end{equation*}
$$

where

$$
A_{i}^{*}(\theta)=\frac{W^{*}(\theta) f_{i}^{*}(\theta) k^{*}(\theta)^{i-5}}{g_{1}^{*}(\theta)^{2}}
$$

Then for $|\varepsilon|>0$ sufficiently small the perturbed systems (4) have a limit cycle bifurcating from the periodic orbit $r\left(\theta, r_{0, s}\right)=k^{*}(\theta) r_{0, s}$ of the period annulus of the center (3) if the first order average function is non-zero. In particular, the polynomial (7) has at most 5 positive simple real roots and they are reached.

Theorem 1 is proved in Section 3 and Theorem 2 is proved in Section 4. In Section 5 we provide two examples that illustrate Theorems 1 and 2 with the maximum number of limit cycles.

It follows of Theorems 1 and 2 the next result.
Corollary 3. For $|\varepsilon|$ sufficiently small the perturbed system (4) has at most 6 limit cycles bifurcating from the periodic orbits of the center (3) using the averaging theory of first order and they are reached.

## 2. Preliminaries

In this section we give some well known results that we shall need for proving Theorem 1.

Consider a system given by

$$
\begin{equation*}
\dot{\mathrm{x}}=F_{0}(t, \mathrm{x})+\varepsilon F_{1}(t, \mathrm{x})+\mathcal{O}\left(\varepsilon^{2}\right) \tag{8}
\end{equation*}
$$

where $\varepsilon \neq 0$ is sufficiently small and the functions $F_{0}, F_{1}: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^{n}$ and $F_{2}: \mathbb{R} \times \Omega \times\left(-\varepsilon_{0}, \varepsilon_{0}\right) \rightarrow \mathbb{R}^{n}$ are $\mathcal{C}^{2}$ functions, $T$-periodic in the first variable and $\Omega$ is an open subset of $\mathbb{R}^{n}$. We suppose that the unperturbed system

$$
\begin{equation*}
\dot{\mathrm{x}}=F_{0}(t, \mathrm{x}) \tag{9}
\end{equation*}
$$

has a submanifold of periodic solutions of dimension $n$.
Let $x(t, z, \varepsilon)$ be the solution of system (9) such that $x(0, z, \varepsilon)=z$. The linearization of the $u$ nperturbed system along a periodic solution $\mathrm{x}(t, \mathrm{z}, 0)$ is

$$
\begin{equation*}
\dot{\mathrm{y}}=D_{\mathrm{x}} F_{0}(t, \mathrm{x}(t, \mathrm{z}, 0)) \mathrm{y} \tag{10}
\end{equation*}
$$

In what follows we denote by $M_{z}(t)$ the fundamental matrix of the linearized system (10) such that $M_{\mathrm{z}}(0)$ is the identity matrix.

We assume that there is an open set $U$ with $\mathrm{Cl}(U) \subset \Omega$ such that for each $z \in \mathrm{Cl}(U), \mathrm{x}(t, \mathrm{z}, 0)$ is $T$-periodic, where $\mathrm{x}(t, z, 0)$ denotes the solution of the unperturbed system (9). We denote by $\mathrm{Cl}(U)$ the closure of $U$. The set $\mathrm{Cl}(U)$ is isochronous for system (9), i.e. it is formed only by periodic orbits with period $T$.

The following result is the a version of averaging theorem for studying the bifurcation of $T$-periodic solutions of system (8) from the periodic solutions $\mathrm{x}(t, \mathrm{z}, 0)$ contained in $\mathrm{Cl}(U)$ of system (9) when $|\varepsilon|>0$ is sufficiently small.

See [4] for a proof. For more details on the averaging theory see [5] and the book [12].

Theorem 4 (Perturbations of an isochronous set). We assume that there exists an open and bounded set $U$ with $\mathrm{Cl}(U) \subset \Omega$ such that for each $\mathrm{z} \in \mathrm{Cl}(U)$, the solution $\mathrm{x}(r, \mathrm{z}, 0)$ is $T$-periodic. Consider the function $\mathcal{F}: \mathrm{Cl}(U) \rightarrow \mathbb{R}^{n}$

$$
\begin{equation*}
\mathcal{F}(\mathrm{z})=\frac{1}{T} \int_{0}^{T} M_{\mathrm{z}}^{-1}(t) F_{1}(t, \mathrm{x}(t, \mathrm{z}, 0)) d t \tag{11}
\end{equation*}
$$

Then if there exists $\mathbf{a} \in U$ with $\mathcal{F}(\mathbf{a})=0$ and $\operatorname{det}((\partial \mathcal{F} / \partial \mathrm{z})(\mathbf{a})) \neq 0$ then there exists a $T$-periodic solution $\mathrm{x}(t, \varepsilon)$ of system (8) such that $\mathrm{x}(0, \varepsilon) \rightarrow \mathbf{a}$ when $\varepsilon \rightarrow 0$.

In fact, if $\mathrm{x}(t, \mathrm{z}, \varepsilon)$ denotes the solution of the differential system (8) such that $\mathrm{x}(0, \mathrm{z}, \varepsilon)=\mathrm{z}$, then the average function satisfies that $\mathrm{x}(T, \mathrm{z}, \varepsilon)-\mathrm{z}=$ $\varepsilon \mathcal{F}(\mathrm{z})+O\left(\varepsilon^{2}\right)$, see for more details [4, 7]. Then, by the Implicit Function Theorem it follows that if $\mathcal{F}(z) \neq 0$, then the simple zeros of the function $\mathcal{F}(z)$ provide limit cycles of the differential system (8).

The following result is the generalized Descartes Theorem about the number of zeros of a real polynomial. See [2] for a proof.

Theorem 5. Consider the real polynomial $p(x)=a_{i_{1}} x^{i_{1}}+a_{i_{2}} x^{i_{2}}+\ldots+a_{i_{r}} x^{i_{r}}$ with $0 \leq i_{1}<i_{2}<\ldots<i_{r}$ and $a_{i_{j}} \neq 0$ real constants for $j \in\{1,2, \ldots, r\}$. When $a_{i_{j}} a_{i_{j+1}}<0$, we say that $a_{i_{j}}$ and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is $m$, then $p(x)$ has at most $m$ positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r-1$ positive real roots.

## 3. Proof of Theorem 1

Suppose that $d \neq 0$. We apply the affine change of variables

$$
\tilde{x}=\alpha x, \quad \tilde{y}=\frac{\alpha^{3} b}{d} y, \quad \tilde{t}=\frac{d}{\alpha^{2}} t,
$$

with $\alpha \neq 0$ and system (3) becomes

$$
\begin{aligned}
& \dot{x}=P(x, y)=-\frac{1}{3} x^{3}+y \\
& \dot{y}=Q(x, y)=a_{1} x^{5}+x^{2} y,
\end{aligned}
$$

where $a_{1}=-\left(4+b^{2}\right) / 12$ and $b \neq 0$. In the case $b=0$ working in a similar way we also can reach the previous differential system. The perturbed system corresponding to the previous system is given by

$$
\begin{align*}
& \dot{x}=-\frac{1}{3} x^{3}+y+\varepsilon p(x, y),  \tag{12}\\
& \dot{y}=a_{1} x^{5}+x^{2} y+\varepsilon q(x, y),
\end{align*}
$$

We write system (12) in the generalized polar coordinates $x=r \cos \theta, y=$ $r^{3} \sin \theta$, and we obtain the differential equation

$$
\begin{equation*}
\frac{d r}{d \theta}=F_{0}(r, \theta)+\varepsilon F_{1}(r, \theta)+\mathcal{O}\left(\varepsilon^{2}\right), \tag{13}
\end{equation*}
$$

in the standard form for applying the averaging theory of first order described in Section 2, where

$$
\begin{aligned}
F_{0}(r, \theta)= & \frac{h_{1}(\theta)}{g_{1}(\theta)} r, \\
F_{1}(r, \theta)= & \frac{144\left(\cos ^{2} \theta+3 \sin ^{2} \theta\right)}{r^{7} g_{1}(\theta)^{2}}\left(Q\left(r \cos \theta, r^{3} \sin \theta\right) p\left(r \cos \theta, r^{3} \sin \theta\right)\right. \\
& \left.-P\left(r \cos \theta, r^{3} \sin \theta\right) q\left(r \cos \theta, r^{3} \sin \theta\right)\right), \\
h_{1}(\theta)= & \cos \theta\left(\left(b^{2}+4\right) \sin \theta \cos ^{4} \theta-6 \sin \theta(\sin (2 \theta)+2)+4 \cos ^{3} \theta\right), \\
g_{1}(\theta)= & \left(4+b^{2}\right) \cos ^{6} \theta-24 \cos ^{3} \theta \sin \theta+36 \sin ^{2} \theta .
\end{aligned}
$$

Note that the differential equation (13) satisfies the assumptions of Theorem 4. Consider $r\left(\theta, r_{0}\right)$ the periodic solution of the differential equation

$$
\frac{d r}{d \theta}=r \frac{h_{1}(\theta)}{g_{1}(\theta)},
$$

such that $r\left(0, r_{0}\right)=r_{0}$. By solving the previous differential equation we get

$$
r\left(\theta, r_{0}\right)=k(\theta) r_{0}
$$

where

$$
k(\theta)=\frac{2^{5 / 6}\left(4+b^{2}\right)^{1 / 6}}{B(\theta)},
$$

with

$$
\begin{aligned}
B(\theta)= & \left(3\left(5 b^{2}-172\right) \cos (2 \theta)+6\left(b^{2}+4\right) \cos (4 \theta)+\left(b^{2}+4\right) \cos (6 \theta)+10 b^{2}\right. \\
& -192 \sin (2 \theta)-96 \sin (4 \theta)+616)^{1 / 6} .
\end{aligned}
$$

Solving the variational equation (10) for our differential equation (13) we obtain that the fundamental matrix $M(\theta)$ is $(k(\theta))$. Using the polynomials $p$ and $q$ given in (5) and system (3) we have that the integrant of the integral (11) for the differential equation (13) is

$$
\begin{aligned}
M^{-1}(\theta) F_{1}\left(\theta, r\left(\theta, r_{0}\right)\right) & =\sum_{i=0}^{17} \frac{W(\theta) f_{i}(\theta)}{g_{1}(\theta)^{2} M(\theta)} r\left(\theta, r_{0}\right)^{i-4} \\
& =\sum_{i=0}^{17} r_{0}^{i-4} \frac{W(\theta) f_{i}(\theta) k(\theta)^{i-5}}{g_{1}(\theta)^{2}} \\
& =\sum_{i=0}^{17} r_{0}^{i-4} A_{i}(\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
& f_{0}(\theta)=4 d_{00} \cos ^{3} \theta-12 d_{00} \sin \theta, \\
& f_{1}(\theta)=4 d_{10} \cos ^{4} \theta-12 d_{10} \sin \theta \cos \theta, \\
& f_{2}(\theta)=\cos ^{5} \theta\left(-b^{2} c_{00}-4 c_{00}+4 d_{20}\right)+12\left(c_{00}-d_{20}\right) \sin \theta \cos ^{2} \theta \text {, } \\
& f_{3}(\theta)=\cos ^{6} \theta\left(-b^{2} c_{10}-4 c_{10}+4 d_{30}\right)+4 \sin \theta \cos ^{3} \theta\left(3 c_{10}+d_{01}-3 d_{30}\right) \\
& -12 d_{01} \sin ^{2} \theta \text {, } \\
& f_{4}(\theta)=\cos ^{7} \theta\left(-b^{2} c_{20}-4 c_{20}+4 d_{40}\right)+4 \sin \theta \cos ^{4} \theta\left(3 c_{20}+d_{11}-3 d_{40}\right) \\
& -12 d_{11} \sin ^{2} \theta \cos \theta \text {, } \\
& f_{5}(\theta)=\sin \theta \cos ^{5} \theta\left(-b^{2} c_{01}-4 c_{01}+12 c_{30}+4 d_{21}-12 d_{50}\right) \\
& +\cos ^{8} \theta\left(-b^{2} c_{30}-4 c_{30}+4 d_{50}\right)+12\left(c_{01}-d_{21}\right) \sin ^{2} \theta \cos ^{2} \theta, \\
& f_{6}(\theta)=\sin \theta \cos ^{6} \theta\left(-b^{2} c_{11}-4 c_{11}+12 c_{40}+4 d_{31}\right)-\left(b^{2}+4\right) c_{40} \cos ^{9} \theta \\
& +4 \sin ^{2} \theta \cos ^{3} \theta\left(3 c_{11}+d_{02}-3 d_{31}\right)-12 d_{02} \sin ^{3} \theta, \\
& f_{7}(\theta)=\sin \theta \cos ^{7} \theta\left(-b^{2} c_{21}-4 c_{21}+12 c_{50}+4 d_{41}\right)-\left(b^{2}+4\right) c_{50} \cos ^{10} \theta \\
& +4 \sin ^{2} \theta \cos ^{4} \theta\left(3 c_{21}+d_{12}-3 d_{41}\right)-12 d_{12} \sin ^{3} \theta \cos \theta \text {, } \\
& f_{8}(\theta)=\sin ^{2} \theta \cos ^{5} \theta\left(-b^{2} c_{02}-4 c_{02}+12 c_{31}+4 d_{22}\right) \\
& -\left(b^{2}+4\right) c_{31} \sin \theta \cos ^{8} \theta+12\left(c_{02}-d_{22}\right) \sin ^{3} \theta \cos ^{2} \theta \text {, } \\
& f_{9}(\theta)=\sin ^{2} \theta \cos ^{6} \theta\left(-b^{2} c_{12}-4 c_{12}+12 c_{41}+4 d_{32}\right) \\
& -\left(b^{2}+4\right) c_{41} \sin \theta \cos ^{9} \theta+4 \sin ^{3} \theta \cos ^{3} \theta\left(3 c_{12}+d_{03}-3 d_{32}\right) \\
& -12 d_{03} \sin ^{4} \theta \text {, } \\
& f_{10}(\theta)=-\left(b^{2}+4\right) c_{22} \sin ^{2} \theta \cos ^{7} \theta+4\left(3 c_{22}+d_{13}\right) \sin ^{3} \theta \cos ^{4} \theta \\
& -12 d_{13} \sin ^{4} \theta \cos \theta \text {, } \\
& f_{11}(\theta)=\sin ^{3} \theta \cos ^{5} \theta\left(-b^{2} c_{03}-4 c_{03}+12 c_{32}+4 d_{23}\right) \\
& -\left(b^{2}+4\right) c_{32} \sin ^{2} \theta \cos ^{8} \theta+12\left(c_{03}-d_{23}\right) \sin ^{4} \theta \cos ^{2} \theta, \\
& f_{12}(\theta)=-\left(b^{2}+4\right) c_{13} \sin ^{3} \theta \cos ^{6} \theta+4\left(3 c_{13}+d_{04}\right) \sin ^{4} \theta \cos ^{3} \theta \\
& -12 d_{04} \sin ^{5} \theta \text {, } \\
& f_{13}(\theta)=-\left(b^{2}+4\right) c_{23} \sin ^{3} \theta \cos ^{7} \theta+4\left(3 c_{23}+d_{14}\right) \sin ^{4} \theta \cos ^{4} \theta \\
& -12 d_{14} \sin ^{5} \theta \cos \theta \text {, } \\
& f_{14}(\theta)=12 c_{04} \sin ^{5} \theta \cos ^{2} \theta-\left(b^{2}+4\right) c_{04} \sin ^{4} \theta \cos ^{5} \theta \text {, } \\
& f_{15}(\theta)=-\left(b^{2}+4\right) c_{14} \sin ^{4} \theta \cos ^{6} \theta+4\left(3 c_{14}+d_{05}\right) \sin ^{5} \theta \cos ^{3} \theta \\
& -12 d_{05} \sin ^{6} \theta \text {, } \\
& f_{16}(\theta)=0, \\
& f_{17}(\theta)=12 c_{05} \sin ^{6} \theta \cos ^{2} \theta-\left(b^{2}+4\right) c_{05} \sin ^{5} \theta \cos ^{5} \theta, \\
& W(\theta)=12\left(3 \sin ^{2} \theta+\cos ^{2} \theta\right) \text {. }
\end{aligned}
$$

Computing the integral (11) we obtain

$$
\mathcal{F}\left(r_{0}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} M^{-1}(\theta) F_{1}\left(\theta, r\left(\theta, r_{0}\right)\right) d \theta=\frac{1}{2 \pi} \sum_{i=0}^{17} r_{0}^{i-4} \int_{0}^{2 \pi} A_{i}(\theta) d \theta
$$

where the function $A_{i}(\theta)$ is defined in the statement of Theorem 1.
If $i$ is even then it is easy to check that $f_{i}(\theta)=-f_{i}(\theta+\pi)$, for $i=0, \ldots, 17$, and $\theta \in[\pi, 3 \pi / 2] \cup[3 \pi / 2, \pi]$. Since that $k(\theta)=k(\theta+\pi), g_{1}(\theta)=g_{1}(\theta+\pi)$ and $W(\theta)=W(\theta+\pi)$, for $\theta \in[\pi, 3 \pi / 2] \cup[3 \pi / 2, \pi]$ we can easily show that

$$
\begin{aligned}
\int_{\pi}^{\frac{3 \pi}{2}} A_{i}(\theta) d \theta & =\int_{\pi}^{\frac{3 \pi}{2}} \frac{f_{i}(\theta) W(\theta) k(\theta)^{i-5}}{g_{1}(\theta)^{2}} d \theta \\
& =\int_{0}^{\frac{\pi}{2}} \frac{f_{i}(\theta+\pi) W(\theta+\pi) k(\theta+\pi)^{i-5}}{g_{1}(\theta+\pi)^{2}} d \theta \\
& =\int_{0}^{\frac{\pi}{2}}-\frac{f_{i}(\theta) W(\theta) k(\theta)^{i-5}}{g_{1}(\theta)^{2}} d \theta \\
& =-\int_{0}^{\frac{\pi}{2}} A_{i}(\theta) d \theta \\
& =\int_{\frac{\pi}{2}}^{\pi} \frac{f_{i}(\theta+\pi) W(\theta+\pi) k(\theta+\pi)^{i-4}}{g_{1}(\theta+\pi)^{2} M(\theta+\pi)} d \theta \\
\int_{\frac{3 \pi}{2}}^{2 \pi} A_{i}(\theta) d \theta & =\int_{\frac{3 \pi}{2}}^{2 \pi} \frac{f_{i}(\theta) W(\theta) k(\theta)^{i-4}}{g_{1}(\theta)^{2} M(\theta)} d \theta \\
& =\int_{\frac{\pi}{2}}^{\pi}-\frac{f_{i}(\theta) W(\theta) k(\theta)^{i-4}}{g_{1}(\theta)^{2} M(\theta)} d \theta \\
& =-\int_{\frac{\pi}{2}}^{\pi} A_{i}(\theta) d \theta
\end{aligned}
$$

Thus if $i$ is even we conclude that

$$
\int_{0}^{2 \pi} A_{i}(\theta) d \theta=0
$$

The coefficients $A_{1}$ and $A_{17}$ are

$$
\begin{aligned}
A_{1} & =\left[7682^{2 / 3} d_{10} \cos \theta(\cos (2 \theta)-2)(-12 \sin \theta+3 \cos \theta+\cos (3 \theta))\right] / L \\
A_{17} & =\left[3 \times 2^{22}\left(b^{2}+4\right)^{2} c_{05} \sin ^{5} \theta \cos ^{2} \theta(\cos (2 \theta)-2) M\right] / N
\end{aligned}
$$

where

$$
\begin{aligned}
L= & \left(b^{2}+4\right)^{2 / 3}\left(3\left(5 b^{2}-172\right) \cos (2 \theta)+6\left(b^{2}+4\right) \cos (4 \theta)\right. \\
& \left.+\left(b^{2}+4\right) \cos (6 \theta)+10 b^{2}-192 \sin (2 \theta)-96 \sin (4 \theta)+616\right)^{4 / 3} \\
M= & \left(b^{2}+4\right) \cos ^{3} \theta-12 \sin \theta \\
N= & \left(3\left(5 b^{2}-172\right) \cos (2 \theta)+6\left(b^{2}+4\right) \cos (4 \theta)+\left(b^{2}+4\right) \cos (6 \theta)\right. \\
& \left.+10 b^{2}-192 \sin (2 \theta)-96 \sin (4 \theta)+616\right)^{4}
\end{aligned}
$$

Computing the integrals of the coefficients $A_{1}$ and $A_{17}$ in the variable $\theta$, in the interval $[0,2 \pi]$ we obtain that both are zero.
Claim: For $i=3,5,7,9,11,13$ or 15 we can choose the parameters that appear in $A_{i}$ such that $\int_{0}^{2 \pi} A_{i}(\theta) d \theta \neq 0$.

Proof. The proof of this claim follows from Example 1.

In summary the function $\mathcal{F}$ defined in (11) can be written as

$$
\begin{equation*}
\mathcal{F}\left(r_{0}\right)=\frac{1}{2 \pi} \sum_{k=1}^{7} r_{0}^{2 k-3} \int_{0}^{2 \pi} A_{2 k+1}(\theta) d \theta . \tag{14}
\end{equation*}
$$

Note that the coefficients $A_{2 k+1}(\theta)$ in (14) are linearly independent for $k=1, . ., 7$. Thus by the generalized Descartes Theorem, the average function $\mathcal{F}$ has at most 6 positive simple zeros which provide limit cycles of system (4), when the average function is non-zero.

## 4. Proof of Theorem 2

Suppose that $d=0$. We take the affine change of coordinates

$$
\tilde{x}=x, \quad \tilde{y}=y \sqrt{-b / c}, \quad \tilde{t}=t \sqrt{-b c},
$$

and system (3) becomes $\dot{x}=-y, \dot{y}=x^{5}$. We write system (4) in the generalized polar coordinates $x=r \cos \theta, y=r^{3} \sin \theta$, and we obtain the differential equation

$$
\begin{equation*}
\frac{d r}{d \theta}=F_{0}(r, \theta)+\varepsilon F_{1}(r, \theta)+\mathcal{O}\left(\varepsilon^{2}\right) \tag{15}
\end{equation*}
$$

in the standard form for applying the averaging theory of first order described in Section 2, where

$$
\begin{aligned}
F_{0}(r, \theta)= & \frac{r\left(\sin \theta \cos ^{5} \theta-\sin \theta \cos \theta\right)}{\cos ^{6} \theta-3 \cos ^{2} \theta+3} \\
F_{1}(r, \theta)= & -\frac{\cos (2 \theta)-2}{r^{4}\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2}}\left(r^{2} \cos ^{5} \theta p\left(r \cos \theta, r^{3} \sin \theta\right)\right. \\
& \left.+\sin \theta q\left(r \cos \theta, r^{3} \sin \theta\right)\right) .
\end{aligned}
$$

Denote by $g_{1}^{*}(\theta)=\cos ^{6} \theta-3 \cos ^{2} \theta+3$. Note that the differential equation (15) satisfies the assumptions of Theorem 4. Consider $r\left(\theta, r_{0}\right)$ the periodic solution of the differential equation $\dot{r}=F_{0}(r, \theta)$ such that $r\left(0, r_{0}\right)=r_{0}$. For solving this differential equation we take $z=\cos ^{2} \theta$ in $g_{1}^{*}(\theta)$, and we obtain a polynomial of degree 3 in $z$ which can be factorized in the form

$$
g_{1}^{*}(z)=\left(z-z_{1}\right)\left(z-z_{2}\right)\left(z-z_{3}\right),
$$

where the coefficients of $g_{2}^{*}(z)=\left(z-z_{1}\right)$ and $g_{3}^{*}(z)=\left(z-z_{2}\right)\left(z-z_{3}\right)$ are reals, and $z_{i}$ are the roots of $g_{1}^{*}$, for $i=1, \ldots, 3$ given by

$$
\begin{aligned}
& z_{1}=-\frac{2+\sqrt[3]{2}(3-\sqrt{5})^{2 / 3}}{2^{2 / 3} \sqrt[3]{3-\sqrt{5}}} \\
& z_{2,3}=\frac{2 \sqrt[3]{2}(1 \mp i \sqrt{3})+(1 \pm i \sqrt{3})(6-2 \sqrt{5})^{2 / 3}}{4 \sqrt[3]{3-\sqrt{5}}}
\end{aligned}
$$

Thus the differential equation (15) with $\varepsilon=0$ can be rewritten into the form

$$
\begin{equation*}
\frac{d r}{d \theta}=r\left(C_{1} \frac{\cos \theta \sin \theta}{-z_{3}+\cos ^{2} \theta}+C_{2} \frac{\cos \theta \sin \theta}{-z_{1}+\cos ^{2} \theta}+C_{3} \frac{\cos \theta \sin \theta}{-z_{2}+\cos ^{2} \theta}\right), \tag{16}
\end{equation*}
$$

where

$$
C_{1}=\frac{z_{3}^{2}-1}{\left(z_{3}-z_{1}\right)\left(z_{3}-z_{2}\right)}, C_{2}=\frac{z_{1}^{2}-1}{\left(z_{1}-z_{2}\right)\left(z_{1}-z_{3}\right)}, C_{3}=\frac{z_{2}^{2}-1}{\left(z_{2}-z_{1}\right)\left(z_{2}-z_{3}\right)} .
$$

The solution of differential equation (16) with initial condition $r\left(0, r_{0}\right)=$ $r_{0}$ is

$$
r\left(\theta, r_{0}\right)=r_{0} k^{*}(\theta),
$$

where

$$
\begin{aligned}
k^{*}(\theta)= & r_{0}\left(1-z_{3}\right)^{C_{1} / 2}\left(1-z_{1}\right)^{C_{2} / 2}\left(1-z_{2}\right)^{C_{3} / 2}\left(\cos ^{2} \theta-z_{3}\right)^{-C_{1} / 2} \\
& \left(\cos ^{2} \theta-z_{1}\right)^{-C_{2} / 2}\left(\cos ^{2} \theta-z_{2}\right)^{-C_{3} / 2} .
\end{aligned}
$$

Solving the variational equation (10) for our differential equation (15) we get that the fundamental matrix is the function $M^{*}(\theta)=\left(k^{*}(\theta)\right)$. Note that $M^{*}(\theta)$ does not depend on $r_{0}$. Using the polynomials $p$ and $q$ given in
(5) and system (3) we have that the integrant of the integral (11) for the differential equation (15) is

$$
\begin{aligned}
M^{*-1}(\theta) F_{1}\left(\theta, r\left(\theta, r_{0}\right)\right) & =\sum_{i=0}^{17} \frac{W^{*}(\theta) f_{i}^{*}(\theta)}{g_{1}^{*}(\theta)^{2} M^{*}(\theta)} r\left(\theta, r_{0}\right)^{i-4} \\
& =\sum_{i=0}^{17} r_{0}^{i-4} \frac{W^{*}(\theta) f_{i}^{*}(\theta) k^{*}(\theta)^{i-5}}{g_{1}^{*}(\theta)^{2}} \\
& =\sum_{i=0}^{17} r_{0}^{i-4} A_{i}^{*}(\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
f_{0}^{*}(\theta) & =d_{00} \sin \theta, \\
f_{1}^{*}(\theta) & =d_{10} \sin \theta \cos \theta, \\
f_{2}^{*}(\theta) & =c_{00} \cos ^{5} \theta+d_{20} \sin \theta \cos ^{2} \theta, \\
f_{3}^{*}(\theta) & =c_{10} \cos ^{6} \theta+d_{01} \sin ^{2} \theta+d_{30} \sin \theta \cos ^{3} \theta, \\
f_{4}^{*}(\theta) & =c_{20} \cos ^{7} \theta+d_{11} \sin ^{2} \theta \cos \theta+d_{40} \sin \theta \cos ^{4} \theta, \\
f_{5}^{*}(\theta) & =\left(c_{01}+d_{50}\right) \sin \theta \cos ^{5} \theta+c_{30} \cos ^{8} \theta+d_{21} \sin ^{2} \theta \cos ^{2} \theta, \\
f_{6}^{*}(\theta) & =c_{11} \sin \theta \cos ^{6} \theta+c_{40} \cos ^{9} \theta+d_{02} \sin ^{3} \theta+d_{31} \sin ^{2} \theta \cos ^{3} \theta, \\
f_{7}^{*}(\theta) & =c_{21} \sin \theta \cos ^{7} \theta+c_{50} \cos ^{10} \theta+d_{12} \sin ^{3} \theta \cos \theta+d_{41} \sin ^{2} \theta \cos ^{4} \theta, \\
f_{8}^{*}(\theta) & =c_{02} \sin ^{2} \theta \cos ^{5} \theta+c_{31} \sin \theta \cos ^{8} \theta+d_{22} \sin ^{3} \theta \cos ^{2} \theta, \\
f_{9}^{*}(\theta) & =c_{12} \sin ^{2} \theta \cos ^{6} \theta+c_{41} \sin \theta \cos ^{9} \theta+d_{03} \sin ^{4} \theta+d_{32} \sin ^{3} \theta \cos ^{3} \theta, \\
f_{10}^{*}(\theta) & =c_{22} \sin ^{2} \theta \cos ^{7} \theta+d_{13} \sin 4 \theta \cos ^{4} \theta, \\
f_{11}^{*}(\theta) & =c_{03} \sin ^{3} \theta \cos ^{5} \theta+c_{32} \sin ^{2} \theta \cos ^{8} \theta+d_{23} \sin ^{4} \theta \cos ^{2} \theta, \\
f_{12}^{*}(\theta) & =c_{13} \sin ^{3} \theta \cos ^{6} \theta+d_{04} \sin ^{5} \theta, \\
f_{13}^{*}(\theta) & =c_{23} \sin ^{3} \theta \cos ^{7} \theta+d_{14} \sin ^{5} \theta \cos \theta, \\
f_{14}^{*}(\theta) & =c_{04} \sin ^{4} \theta \cos ^{5} \theta, \\
f_{15}^{*}(\theta) & =c_{14} \sin ^{4} \theta \cos ^{6} \theta+d_{05} \sin ^{6} \theta, \\
f_{16}^{*}(\theta) & =0, \\
f_{17}^{*}(\theta) & =c_{05} \sin ^{5} \theta \cos ^{5} \theta, \\
W^{*}(\theta) & =3 \sin ^{2} \theta+\cos ^{2} \theta .
\end{aligned}
$$

Computing the integral (11) we obtain

$$
\mathcal{F}^{*}\left(r_{0}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(M^{*}\right)^{-1}(\theta) F_{1}\left(\theta, r\left(\theta, r_{0}\right)\right) d \theta=\frac{1}{2 \pi} \sum_{i=0}^{17} r_{0}^{i-4} \int_{0}^{2 \pi} A_{i}^{*}(\theta) d \theta
$$

where the function $A_{i}^{*}(\theta)$ is defined in the statement of Theorem 4.

Analogously as in the proof of Theorem 1 we can show that if $i$ is even then

$$
\int_{0}^{2 \pi} A_{i}^{*}(\theta) d \theta=0
$$

The coefficients $A_{1}^{*}, A_{13}^{*}$ and $A_{17}^{*}$ are given by

$$
\begin{aligned}
A_{1}^{*} & =-d_{10} \frac{2^{\frac{55}{9}}(7-3 \sqrt{5})^{4 / 9} \sin (2 \theta)(\cos (2 \theta)-2)}{(3-\sqrt{5})^{8 / 9}(-33 \cos (2 \theta)+6 \cos (4 \theta)+\cos (6 \theta)+58)^{4 / 3}} \\
A_{13}^{*} & =-\frac{2^{\frac{4}{3}}(2+\sqrt{5})^{4 / 9} \sin ^{3} \theta \cos \theta(\cos (2 \theta)-2)\left(c_{23} \cos ^{6} \theta+d_{14} \sin ^{2} \theta\right)}{(1+\sqrt{5})^{4 / 3}\left(\cos ^{6} \theta-3 \cos ^{2} \theta+3\right)^{4 / 3}\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2}}, \\
A_{17}^{*} & =-c_{05} \frac{32 \sin ^{5} \theta \cos ^{5} \theta(\cos (2 \theta)-2)}{R(\theta)},
\end{aligned}
$$

where

$$
\begin{aligned}
R(\theta)= & \sqrt[3]{47+21 \sqrt{5}}\left(2 \sqrt[3]{2} \cos ^{4} \theta+4 \sqrt[3]{3\left(2+2^{2 / 3} \sqrt[3]{3-\sqrt{5}}+2^{2 / 3} \sqrt[3]{3+\sqrt{5}}\right.}\right) \\
& \left.\cos ^{2} \theta+(2(3+\sqrt{5}))^{2 / 3}+(6-2 \sqrt{5})^{2 / 3}+4 \sqrt[3]{2}\right)\left(-2(3-\sqrt{5})^{2 / 3}\right. \\
& \cos ^{4} \theta+\left(2 \sqrt[3]{6-2 \sqrt{5}}-2^{2 / 3}(\sqrt{5}-3)\right) \cos ^{2} \theta+\sqrt[3]{30 \sqrt{5}-50} \\
& \left.+2(3-\sqrt{5})^{2 / 3}-3 \sqrt[3]{6-2 \sqrt{5}}-22^{2 / 3}\right)^{2}\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2} .
\end{aligned}
$$

The integrals of the coefficients $A_{1}^{*}, A_{13}^{*}$ and $A_{17}^{*}$ in the variable $\theta$, in the interval $[0,2 \pi]$ are zero because $A_{1}^{*}, A_{13}^{*}$ and $A_{17}^{*}$ are odd functions.
Claim: For $i=3,5,7,9,11$ or 15 we can choose the parameters that appear in $A_{i}^{*}$ such that $\int_{0}^{2 \pi} A_{i}^{*}(\theta) d \theta \neq 0$.

Proof. The proof of this claim follows from Example 2.
In short the function $\mathcal{F}$ defined in (11) can be written as

$$
\begin{equation*}
\mathcal{F}^{*}\left(r_{0}\right)=\frac{1}{2 \pi}\left(\sum_{k=1}^{5} r_{0}^{2 k-3} \int_{0}^{2 \pi} A_{2 k+1}^{*}(\theta) d \theta+r_{0}^{11} \int_{0}^{2 \pi} A_{15}^{*}(\theta) d \theta\right) \tag{17}
\end{equation*}
$$

Note that the coefficients $A_{2 k+1}^{*}(\theta)$ in (17) are linearly independent for $k=1,2,3,4,5,7$. Thus by the generalized Descartes Theorem, the average function $\mathcal{F}^{*}$ has at most 5 positive simple zeros which provide limit cycles of system (4), when the average function is non-zero.

## 5. Examples

Example 1. Consider the cubic polynomial differential system with a center at the origin

$$
\dot{x}=-\frac{1}{3} x^{3}+y, \quad \dot{y}=-\frac{5}{12} x^{5}+x^{2} y
$$

with the perturbation

$$
\begin{equation*}
\dot{x}=\dot{x}=-\frac{1}{3} x^{3}+y, \quad \dot{y}=-\frac{5}{12} x^{5}+x^{2} y+\varepsilon q(x, y) \tag{18}
\end{equation*}
$$

where

$$
q(x, y)=d_{01} y+d_{21} x^{2} y+d_{12} x y^{2}+d_{03} y^{3}+d_{23} x^{2} y^{3}+d_{14} x y^{4}+d_{05} y^{5}
$$

Writing system (18) in the coordinates $x=r \cos \theta$ and $y=r^{3} \sin \theta$ and taking the quotient $\dot{r} / \dot{\theta}$ we get the following system in the standard form of Theorem 4 for applying the averaging theory

$$
\begin{equation*}
\frac{d r}{d \theta}=F_{0}(r, \theta)+\varepsilon F_{1}(r, \theta)+\mathcal{O}\left(\varepsilon^{2}\right) \tag{19}
\end{equation*}
$$

where

$$
\begin{aligned}
& F_{0}(r, \theta)=\frac{r \cos \theta\left(-6 \sin \theta(\sin (2 \theta)+2)+4 \cos ^{3} \theta+5 \sin \theta \cos ^{4} \theta\right)}{36 \sin ^{2} \theta+5 \cos ^{6} \theta-24 \sin \theta \cos ^{3} \theta} \\
& F_{1}(r, \theta)=-48 C(\theta) E(\theta) \frac{q\left(r \cos \theta, r^{3} \sin \theta\right)}{r^{4}\left(36 \sin ^{2} \theta+5 \cos ^{6} \theta-24 \sin \theta \cos ^{3} \theta\right)^{2}}
\end{aligned}
$$

with $C(\theta)=\cos (2 \theta)-2$, and $E(\theta)=\cos ^{3} \theta-3 \sin \theta$. Thus for system (19) we have $M(\theta)=(k(\theta))=\left((160 / G(\theta))^{1 / 6}\right)$, where
$G(\theta)=-192 \sin (2 \theta)-96 \sin (4 \theta)-501 \cos (2 \theta)+30 \cos (4 \theta)+5 \cos (6 \theta)+626$,
and the integrant of the integral (11) of system (19) is

$$
\sum_{1}^{7} r_{0}^{2 k-3} A_{2 k+1}(\theta)
$$

with

$$
\begin{aligned}
& A_{3}(\theta)=-d_{01} \frac{3072 \sqrt[3]{\frac{2}{5}} \sin \theta C(\theta)(-12 \sin \theta+3 \cos \theta+\cos (3 \theta))}{T(\theta)^{5 / 3}}, \\
& A_{5}(\theta)=-d_{21} \frac{48 \sin \theta \cos ^{2} \theta C(\theta) E(\theta)}{\left(36 \sin ^{2} \theta+5 \cos ^{6} \theta-24 \sin \theta \cos ^{3} \theta\right)^{2}}, \\
& A_{7}(\theta)=-d_{12} \frac{983042^{2 / 3} \sqrt[3]{5} \sin ^{2} \theta \cos \theta C(\theta) E(\theta)}{T(\theta)^{7 / 3}} \\
& A_{9}(\theta)=-d_{03} \frac{393216 \sqrt[3]{2} 5^{2 / 3} \sin ^{3} \theta C(\theta) E(\theta)}{T(\theta)^{8 / 3}} \\
& A_{11}(\theta)=-d_{23} \frac{7864320 \sin ^{3} \theta \cos ^{2} \theta C(\theta) E(\theta)}{T(\theta)^{3}} \\
& A_{13}(\theta)=-d_{14} \frac{157286402^{2 / 3} \sqrt[3]{5} \sin ^{4} \theta \cos \theta C(\theta) E(\theta)}{T(\theta)^{10 / 3}} \\
& A_{15}(\theta)=-d_{05} \frac{62914560 \sqrt[3]{2} 5^{2 / 3} \sin ^{5} \theta C(\theta) E(\theta)}{T(\theta)^{11 / 3}}
\end{aligned}
$$

and
$T(\theta)=-192 \sin (2 \theta)-96 \sin (4 \theta)-501 \cos (2 \theta)+30 \cos (4 \theta)+5 \cos (6 \theta)+626$.
Computing numerically the integral (11) for system (19) we obtain

$$
\begin{aligned}
\mathcal{F}\left(r_{0}\right)= & \frac{1}{r_{0}}\left(-4.2608 . . d_{01}-2.0944 . . d_{21} r_{0}^{2}-1.2770 . . d_{12} r_{0}^{4}-1.2427 . . d_{03} r_{0}^{6}\right. \\
& \left.-1.0908 . . d_{23} r_{0}^{8}-0.7348 . . d_{14} r_{0}^{10}-0.5419 . . d_{05} r_{0}^{12}\right)
\end{aligned}
$$

Taking

$$
\begin{gathered}
d_{01}=-\frac{720}{4.2608 . .}, \quad d_{21}=\frac{1764}{2.0944 . .}, \quad d_{12}=-\frac{1624}{1.2770 . .} \\
d_{03}=\frac{735}{1.2427 . .}, \quad d_{23}=-\frac{175}{1.0908 . .}, \quad d_{14}=\frac{21}{0.7348 . .}, \quad d_{05}=-\frac{1}{0.5419 . .} .
\end{gathered}
$$

The function $\mathcal{F}$ becomes

$$
\mathcal{F}\left(r_{0}\right)=\frac{r_{0}^{12}-21 r_{0}^{10}+175 r_{0}^{8}-735 r_{0}^{6}+1624 r_{0}^{4}-1764 r_{0}^{2}+720}{r_{0}}=\frac{1}{r_{0}} \prod_{i=1}^{6}\left(r_{0}^{2}-i\right)
$$

Thus we have that $\mathcal{F}$ has 6 positive simple zeros given by $r_{0, i}=\sqrt{i}$, for $i=1, \ldots, 6$ which by Theorem 4 , provide 6 limit cycles of the perturbed system (18) for $\varepsilon \neq 0$ sufficiently small.

Example 2. Consider the cubic polynomial differential system with a center at the origin

$$
\dot{x}=-y, \quad \dot{y}=x^{5}
$$

with the perturbation

$$
\begin{equation*}
\dot{x}=\dot{x}=-y+\varepsilon p(x, y), \quad \dot{y}=x^{5}+\varepsilon q(x, y) \tag{20}
\end{equation*}
$$

where

$$
\begin{aligned}
& p(x, y)=c_{30} x^{3}+c_{50} x^{5}+c_{14} x y^{4} \\
& q(x, y)=d_{01} y+d_{03} y^{3}+d_{23} x^{2} y^{3}
\end{aligned}
$$

Writing system (20) in the coordinates $x=r \cos \theta$ and $y=r^{3} \sin \theta$ and taking the quotient $\dot{r} / \dot{\theta}$ we obtain the following system in the standard form of Theorem 4 for applying the averaging theory

$$
\begin{equation*}
\frac{d r}{d \theta}=F_{0}(r, \theta)+\varepsilon F_{1}(r, \theta)+\mathcal{O}\left(\varepsilon^{2}\right), \tag{21}
\end{equation*}
$$

with $F_{0}(r, \theta)$ given in the proof of Theorem 1 and

$$
\begin{aligned}
F_{1}(r, \theta)= & -\frac{\cos (2 \theta)-2}{r\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2}}\left[r^{6} \sin ^{4} \theta\left(c_{14} r^{6} \cos ^{6} \theta+d_{03}+d_{23} r^{2} \cos ^{2} \theta\right)\right. \\
& \left.+r^{2} \cos ^{8} \theta\left(c_{30}+c_{50} r^{2} \cos ^{2} \theta\right)+d_{01} \sin ^{2} \theta\right] .
\end{aligned}
$$

The functions $k^{*}(\theta)$ and $M^{*}(\theta)$ for system (21) are given also in the proof of Theorem 1 and the integrant of the integral (11) of system (21) is

$$
\sum_{k=1}^{5} r_{0}^{2 k-3} A_{2 k+1}^{*}(\theta)+r_{0}^{11} A_{15}^{*}(\theta)
$$

where

$$
\begin{aligned}
& A_{3}^{*}(\theta)=-d_{01} \frac{256(6-2 \sqrt{5})^{2 / 9} \sin ^{2} \theta(\cos (2 \theta)-2)}{\sqrt[9]{7-3 \sqrt{5}}(-33 \cos (2 \theta)+6 \cos (4 \theta)+\cos (6 \theta)+58)^{5 / 3}}, \\
& A_{5}^{*}(\theta)=-c_{30} \frac{\cos ^{8} \theta(\cos (2 \theta)-2)}{\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2}}, \\
& A_{7}^{*}(\theta)=-c_{50} \frac{\cos ^{10} \theta(\cos (2 \theta)-2)}{\sqrt[3]{\cos ^{6} \theta-3 \cos ^{2} \theta+3}\left(3 \sin ^{2} \theta+\cos ^{6} \theta\right)^{2}}, \\
& A_{9}^{*}(\theta)=-d_{03} \frac{2^{\frac{40}{3} \sin ^{4} \theta(\cos (2 \theta)-2)} \frac{(-33 \cos (2 \theta)+6 \cos (4 \theta)+\cos (6 \theta)+58)^{8 / 3}}{},}{A_{11}^{*}(\theta)=-d_{23} \frac{2^{\frac{43}{3}}(\sqrt{5}-1)^{4 / 3} \sin ^{4} \theta \cos ^{2} \theta(\cos (2 \theta)-2)}{(3-\sqrt{5})^{2 / 3}(-33 \cos (2 \theta)+6 \cos (4 \theta)+\cos (6 \theta)+58)^{3}},} \begin{array}{l}
A_{15}^{*}(\theta)=-c_{14} \frac{2^{\frac{55}{3}} \sin ^{4}(\theta) \cos 6(\cos (2 \theta)-2)}{(-33 \cos (2 \theta)+6 \cos (4 \theta)+\cos (6 \theta)+58)^{11 / 3}},
\end{array},=\text {, },
\end{aligned}
$$

Computing numerically the integral (11) for system (21) we obtain

$$
\begin{aligned}
\mathcal{F}^{*}\left(r_{0}\right)= & \frac{1}{r_{0}}\left(2.1033 . . d_{01}+1.8138 . . c_{30} r_{0}^{2}+1.6169 . . c_{50} r_{0}^{4}+0.6310 . . d_{03} r_{0}^{6}\right. \\
& \left.+0.1512 . . d_{23} r_{0}^{8}+0.0394 . . c_{14} r_{0}^{12}\right) .
\end{aligned}
$$

Taking

$$
\begin{gathered}
d_{01}=-\frac{1800}{2.1033 . .}, \quad c_{30}=\frac{3990}{1.8138 . .}, \quad c_{50}=-\frac{3101}{1.6169 . .} \\
d_{03}=\frac{1050}{0.6310 . .}, \quad d_{23}=-\frac{140}{0.1512 . .}, \quad c_{14}=\frac{1}{0.0394 . .}
\end{gathered}
$$

The function $\mathcal{F}^{*}$ is now given by
$\mathcal{F}^{*}\left(r_{0}\right)=\frac{r_{0}^{12}-140 r_{0}^{8}+1050 r_{0}^{6}-3101 r_{0}^{4}+3990 r_{0}^{2}-1800}{r_{0}}=\frac{r_{0}^{2}+15}{r_{0}} \prod_{i=1}^{5}\left(r_{0}^{2}-i\right)$.
Thus we have that $\mathcal{F}^{*}$ has 5 positive simple zeros given by $r_{0, i}=\sqrt{i}$, for $i=1, \ldots, 5$ which by Theorem 4, provide 5 limit cycles of the perturbed system (20) with $\varepsilon \neq 0$ sufficiently small.

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