Analytic integrability of Hamiltonian systems with a homogeneous polynomial potential of degree 4

Jaume Llibre,[1,a)] Adam Mahdi,[2,3,b)] and Claudia Valls[4,c)]
[1]*Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[2]*Mathematics Department, University of North Carolina at Charlotte, Charlotte, North Carolina 28223, USA*
[3]*Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland*
[4]*Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal*

(Received 4 September 2009; accepted 21 December 2010; published online 25 January 2011)

In the analytic case we prove the conjecture of Maciejewski and Przybylska [J. Math. Phys. **46**(6), 062901 (2005)] regarding Hamiltonian systems with a homogeneous polynomial potential of degree 4. The proof of the conjecture completes the characterization of all the analytic integrable Hamiltonian system with a homogeneous polynomial potential of degree 4. © *2011 American Institute of Physics*. [doi:10.1063/1.3544473]

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

We consider $\mathbb{C}^4$ as a symplectic linear space with canonical variables $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. We are interested in Hamiltonian systems defined by the Hamiltonian function:

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(\mathbf{q}), \tag{1}$$

where $V(\mathbf{q}) = V(q_1, q_2)$ is a homogeneous polynomial of degree k. To be more precise, we consider the following system of four differential equations:

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1,2. \tag{2}$$

Let $A = A(\mathbf{q},\mathbf{p})$ and $B = B(\mathbf{q},\mathbf{p})$ be two functions. Then their *Poisson bracket* $\{A, B\}$ is given by

$$\{A, B\} = \sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

We say that functions A and B *commute* or that they are *in involution* if $\{A, B\} = 0$. We say that a non-constant function $F = F(\mathbf{q},\mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it commutes with the Hamiltonian function H, i.e., $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric, it is clear that H itself is always a first integral. We will say that a 2–degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has two functionally independent first integrals: H, and an additional one F, which are in involution.

In the beginning of 1980's, all integrable Hamiltonian systems (1) with a homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4

[a)]Electronic mail: jllibre@mat.uab.cat.
[b)]Electronic mail: adam.mahdi@uncc.edu.
[c)]Electronic mail: cvalls@math.ist.utl.pt.