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Centers on center manifolds in the Lü system

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ABSTRACT

We confirm a conjecture of Mello and Coelho [Phys. Lett. A 373 (2009) 1116] concerning the existence of centers on local center manifolds at equilibria of the Lü system of differential equations on $\mathbb{R}^3$. Our proof shows that the local center manifolds are algebraic ruled surfaces, and are unique.

1. Introduction

The Lü system is the system of nonlinear ordinary differential equations on $\mathbb{R}^3$ given by

$$\begin{aligned}\dot{x} &= P(x,y,z) = a(y-x),\\ \dot{y} &= Q(x,y,z) = cy - xz,\\ \dot{z} &= R(x,y,z) = -bz + xy,\end{aligned} \tag{1}$$

where a, b, and c are real parameters. This system has an equilibrium at the origin for all values of the parameters and a pair of symmetrically located equilibria $Q_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ when $bc > 0$. For parameter values in the surface $\mathcal{S} = \{(a,b,c)\colon ab > 0,\ c = (a+b)/3\}$ the matrix of first partial derivatives of the right-hand sides of (1) at Q_+ and Q_- has two purely imaginary eigenvalues and one real eigenvalue, which has opposite sign from $a+b$. It was shown in [1] that for parameter values in the straight line $\mathcal{C} = \{(a,b,c)\colon\ a \neq 0,\ b = 2c,\ c = a\}$ in $\mathcal{S}$ the first three Lyapunov quantities at $Q_\pm$ are zero, which led the authors of that letter to conjecture the following result, which they described as unexpected.

Conjecture. *(See [1].) For parameter values in $\mathcal{C}$ satisfying $a+b \neq 0$ the equilibria $Q_\pm$ are nonlinear centers for the flow of system (1) restricted to the local center manifolds.*

In this Letter we prove that Conjecture is true and in the process unfold some of the structure of the local center manifolds.

2. Proof of Conjecture

System (1) is invariant under the involution $(x,y,z) \to (-x,-y,z)$, so we may confine our attention to Q_+. By the Center Manifold Theorem (see for example [2]) there is a two-dimensional invariant manifold W^c in a neighborhood of Q_+ that is tangent to the center eigenspace at Q_+ and contains all the local recurrent behavior. But although system (1) is polynomial, a priori W^c need not be even C^∞, and it need not be unique. Our first attempt at a proof of Conjecture is to try to avoid the center manifold altogether by means of the following theorem of Lyapunov [3]. A proof may be found in [4, §13].

Theorem *(Lyapunov Center Theorem). Suppose U is an open neighborhood of the origin in $\mathbb{R}^3$, $\mathbf{f}: U \to \mathbb{R}^3$ is a real analytic mapping, and that $d\mathbf{f}(\mathbf{0})$ has one non-zero and two pure imaginary eigenvalues. By an invertible linear change of coordinates and a possibly negative rescaling of time place the system of differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ in the form*

$$\begin{aligned}\dot{u} &= P(u,v,w) = -\omega v + \widetilde{P}(u,v,w),\\ \dot{v} &= Q(u,v,s) = \omega u + \widetilde{Q}(u,v,w),\\ \dot{w} &= R(u,v,w) = -\lambda w + \widetilde{R}(u,v,w),\end{aligned} \tag{2}$$

where ω and λ are positive real numbers. Let $\mathfrak{X}$ denote the corresponding vector field

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