Physica D 240 (2011) 1928–1935

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Polynomial integrability of the Hamiltonian systems with homogeneous potential of degree −3

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ARTICLE INFO

Article history:
Received 21 February 2011
Received in revised form
3 September 2011
Accepted 5 September 2011
Available online 14 September 2011
Communicated by V. Rom-Kedar

Keywords:
Hamiltonian system with 2-degrees of freedom
Homogeneous potential of degree −3
Polynomial integrability

ABSTRACT

In this paper, we study the polynomial integrability of natural Hamiltonian systems with two degrees of freedom having a homogeneous potential of degree k given either by a polynomial, or by an inverse of a polynomial. For $k = -2, -1, \ldots, 3, 4$, their polynomial integrability has been characterized. Here, we have two main results. First, we characterize the polynomial integrability of those Hamiltonian systems with homogeneous potential of degree −3. Second, we extend a relation between the nontrivial eigenvalues of the Hessian of the potential calculated at a Darboux point to a family of Hamiltonian systems with potentials given by an inverse of a homogeneous polynomial. This relation was known for such Hamiltonian systems with homogeneous polynomial potentials. Finally, we present three open problems related with the polynomial integrability of Hamiltonian systems with a rational potential.

1. Introduction and statement of the main results

Ordinary differential equations in general and Hamiltonian systems, in particular, play a very important part in many branches of the applied sciences. The question whether a differential model admits a first integral is of fundamental importance as first integrals give conservation laws for the model and that enables us to lower the dimension of the system. Moreover, knowing a sufficient number of first integrals allows to solve the system explicitly. Until the end of the 19th century the majority of scientists thought that the equations of classical mechanics were integrable and finding the first integrals was mainly a problem of computation. In fact, integrability is a rare phenomenon and in general it is very hard to determine whether a given Hamiltonian system is integrable or not.

In this work, we are concerned with the polynomial integrability of the natural Hamiltonian systems defined by the Hamiltonian function

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q_1.q_2), \tag{1}$$

where $V(q_1, q_2) \in \mathbb{C}(q_1, q_2)$ is a rational homogeneous potential of degree k given by either a polynomial or an inverse of a polynomial. Here $\mathbb{C}(q_1, q_2)$ as usual is the field of rational functions over $\mathbb{C}$ in the variables q_1, q_2. To be more precise, we consider the following system of four differential equations

$$\dot{q}_i = p_i, \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2. \tag{2}$$

Let $A = A(\mathbf{q}, \mathbf{p})$ and $B = B(\mathbf{q}, \mathbf{p})$ be two functions, where $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$. We define the *Poisson bracket* of A and B as

$$\{A, B\} = \sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

The functions A and B are *in involution* if $\{A, B\} = 0$. A non-constant function $F = F(\mathbf{q}, \mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it is in involution with the Hamiltonian function H, i.e. $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric, it follows that H itself is always a first integral. A 2-degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has two functionally independent first integrals H and F. As usual, H and F are *functionally independent* if their gradients are linearly independent at all points of $\mathbb{C}^4$ except, perhaps, in a zero Lebesgue set.

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doi:10.1016/j.physd.2011.09.003