Physics Letters A 375 (2011) 1845–1849

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Polynomial integrability of the Hamiltonian systems with homogeneous potential of degree −2

Jaume Llibre [a,*], Adam Mahdi [b,c], Claudia Valls [d]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[b] Mathematics Department, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[c] Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland
[d] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

ARTICLE INFO

Article history:
Received 11 January 2011
Received in revised form 12 March 2011
Accepted 17 March 2011
Available online 31 March 2011
Communicated by A.P. Fordy

Keywords:
Natural Hamiltonian systems with 2-degrees of freedom
Homogeneous polynomial potential

ABSTRACT

We characterize the analytic integrability of Hamiltonian systems with Hamiltonian $H=\frac{1}{2}\sum_{i=1}^{2}p_i^2+V(q_1,q_2)$, having homogeneous potential $V(q_1,q_2)$ of degree -2.

1. Introduction

We consider $\mathbb{C}^4$ as a symplectic linear space with canonical variables $q=(q_1,q_2)$ and $p=(p_1,p_2)$. We are interested in Hamiltonian systems defined by the Hamiltonian function

$$H=\frac{1}{2}\sum_{i=1}^{2}p_i^2+V(q), \tag{1}$$

where $V(q)=V(q_1,q_2)$ is a homogeneous function of degree k. To be more precise we consider the following system of four differential equations

$$\dot{q}_i=p_i, \qquad \dot{p}_i=-\frac{\partial V}{\partial q_i}, \quad i=1,2. \tag{2}$$

Let $A=A(q,p)$ and $B=B(q,p)$ be two functions. Then their *Poisson bracket* $\{A,B\}$ is given by

$$\{A,B\}=\sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right).$$

We say that functions A and B are *in involution* if $\{A,B\}=0$. We say that a non-constant function $F=F(q,p)$ is a *first integral* for the Hamiltonian system (2) if it commutes with the Hamiltonian function H, i.e. $\{H,F\}=0$. Since the Poisson bracket is antisymmetric it is clear that H itself is always a first integral. We say that a 2-degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has 2 functionally independent first integrals: H, and an additional one F, which are in involution. As usual H and F are *functionally independent* if their gradients are linearly independent at all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue set.

First we recall basic properties of system (2). Let $\mathrm{PO}_2(\mathbb{C})$ denote the group of 2×2 complex matrices A such that $AA^T=\alpha I$, where I is the identity matrix and $\alpha\in\mathbb{C}\setminus\{0\}$. We say that potentials $V_1(q)$ and $V_2(q)$ are *equivalent* if there exists a matrix $A\in\mathrm{PO}_2(\mathbb{C})$ such that $V_1(q)=V_2(Aq)$. So we divide all potentials into equivalent classes. Here a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple lemma. For a proof see [8].

Lemma 1. *Let V_1 and V_2 be two equivalent potentials. If Hamiltonian system (2) is integrable with potential V_1 then it is also integrable with V_2.*

In the beginning of 80's all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4 in the variables p_1 and p_2 were found, see [14,5,3,6,2] and also [7] for the list of corresponding additional first integrals. We remark that all these first integrals are polynomials in the variables p_1, p_2, q_1

* Corresponding author. Tel.: +34 935811303; fax: +34 935812790.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), mahdi@mat.uab.cat, adam.mahdi@uncc.edu (A. Mahdi), cvalls@math.ist.utl.pt (C. Valls).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2011.03.042