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PHASE PORTRAITS AND INVARIANT STRAIGHT LINES OF CUBIC POLYNOMIAL VECTOR FIELDS HAVING A QUADRATIC RATIONAL FIRST INTEGRAL

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ABSTRACT. In this paper we classify all cubic polynomial differential systems having a rational first integral of degree two. In other words we characterize all the global phase portraits of the cubic polynomial differential systems having all their orbits contained in conics. We also determine their configurations of invariant straight lines. We show that there are exactly 38 topologically different phase portraits in the Poincaré disc associated with this family of cubic polynomial differential systems up to a reversed sense of their orbits.

1. Introduction and statement of the main results. Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system or a vector field defined on the plane $\mathbf{R}^2$, the existence of a first integral determines completely its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a vector field on* $\mathbf{R}^2$*, how to recognize if this vector field has a first integral*? One of the easiest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields which are not Hamiltonian are, in general, very difficult to detect. In this paper we will characterize the cubic polynomial vectors fields having a rational first integral of degree 2.

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