# ON THE NUMBER OF LIMIT CYCLES OF THE DIFFERENTIAL EQUATION 

$$
\frac{d r}{d \theta}=\frac{a(\theta)}{\sum_{j=0}^{n} a_{j}(\theta) r^{j}}
$$

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#### Abstract

The notion of Hilbert number from polynomial differential systems in the plane of degree $n$ can be extended to the differential equations of the form $$
\begin{equation*} \frac{d r}{d \theta}=\frac{a(\theta)}{\sum_{j=0}^{n} a_{j}(\theta) r^{j}} \tag{*} \end{equation*}
$$ defined in the region of the cylinder where $\sum_{j=0}^{n} a_{j}(\theta) r^{j} \neq 0$ as follows. The Hilbert number $\mathbb{H}(n)$ is the supremum of the number of limit cycles that any differential equation $\left(^{*}\right.$ ) on the cylinder of degree $n$ in the variable $r$ can have. We prove that $\mathbb{H}(n)=\infty$ for all $n \geq 1$.


## 1. Introduction

In the article [6] Lins Neto studied the following problem posed by Charles Pugh.

Problem 1. Let $a_{0}, a_{1}, \ldots, a_{n}: \mathbb{S}^{1} \rightarrow \mathbb{R}$ be continuous $2 \pi$-periodic functions and consider the differential equation

$$
\begin{equation*}
\frac{d r}{d \theta}=a_{0}(\theta)+a_{1}(\theta) r+\ldots+a_{n}(\theta) r^{n} \tag{1}
\end{equation*}
$$

on the cylinder $(\theta, r) \in \mathbb{S}^{1} \times \mathbb{R}$. Then the problem is to know the number of isolated periodic solutions (i.e. limit cycles) of the differential equation (1) in function of $n$.

Problem 1 was motivated by the Hilbert's 16-th problem (see for instance $[3,4,5]$ ), because some polynomial differential systems in the

[^0]plane can be reduced to equations (1) as all the polynomial differential systems of degree 2 (see for instance the proposition of [6]), all polynomial differential systems with the linear center $\dot{x}=-y, \dot{y}=x$ with nonlinearities given by homogeneous polynomials of degree $n$ for all positive integer $n$ (see for instance [7]), all polynomial differential systems such that in polar coordinates $(r, \theta)$ have $\dot{\theta}=1, \ldots$ See also [1] for more details on the differential equations (1).

For polynomial differential systems in the plane it is defined the Hilbert number $H(n)$, i.e. the supremum of the number of limit cycles that a polynomial differential system in the plane of degree $n$ can have. For the moment it is unknown if the Hilbert number is finite or infinite when $n>1$. We can extend the notion of Hilbert number to the differential equations (1) defined on the cylinder as follows. The Hilbert number $\mathcal{H}(n)$ is the supremum of the number of limit cycles that a differential equation (1) on the cylinder of degree $n$ in the variable $r$ can have.

The Hilbert number for the Problem 1 has the following answer. For the differential equations of the form
(i) $\frac{d r}{d \theta}=a_{0}(\theta)+a_{1}(\theta) r$ (periodic linear differential equations) it is known that $\mathcal{H}(1)=1$.
(ii) $\frac{d r}{d \theta}=a_{0}(\theta)+a_{1}(\theta) r+a_{2}(\theta) r^{2}$ (periodic Riccati differential equations) we have that $\mathcal{H}(2)=2$, see for instance Theorem 1 of [6].
(iii) $\frac{d r}{d \theta}=a_{0}(\theta)+a_{1}(\theta) r+a_{2}(\theta) r^{2}+a_{3}(\theta) r^{3}$ (periodic Abel differential equations) can have $k$ limit cycles for all positive $k$, see the example of section 3 of [6]. So $\mathcal{H}(3)=\infty$.
(iv) $\frac{d r}{d \theta}=a_{0}(\theta)+a_{1}(\theta) r+\ldots+a_{s}(\theta) r^{s}$ can have $k$ limit cycles for all positive $k$. We have the same conclusion than for the periodic Abel differential equation and the proof follows easily modifying the proof of (iii). Hence $\mathcal{H}(n)=\infty$ for $n>3$.

In this paper we consider the following problem:
Problem 2. Let $a, a_{0}, a_{1}, \ldots, a_{n}: \mathbb{S}^{1} \rightarrow \mathbb{R}$ be continuous $2 \pi$-periodic functions and consider the differential equation

$$
\begin{equation*}
\frac{d r}{d \theta}=\frac{a(\theta)}{a_{0}(\theta)+a_{1}(\theta) r+\ldots+a_{n}(\theta) r^{n}}, \tag{2}
\end{equation*}
$$

on the region of the cylinder $(\theta, r) \in \mathbb{S}^{1} \times \mathbb{R}$ where the denominator of (2) does not vanish. Then the problem is to know the number of limit cycles of the differential equation (2) in function of $n$.

Again we can extend the notion of Hilbert number to the differential equations (2) defined on the cylinder as follows. The Hilbert number $\mathbb{H}(n)$ is the supremum of the number of limit cycles that a differential equation (2) on the cylinder of degree $n$ in the variable $r$ can have.

The main result of this paper is to compute the Hilbert number for the Problem 2.

Theorem 1.1. For all positive integer $k$ there are differential equations (2) with $n=1$ having at least $k$ limit cycles. So $\mathbb{H}(1)=\infty$.

Theorem 1.1 is proved in section 3 using the averaging theory of first order for studying the periodic solutions. We present the results of this theory that we need in section 2.

A corollary of Theorem 1.1 is the following.
Corollary 1.2. For all positive integers $n$ and $k$ there are differential equations (2) having at least $k$ limit cycles. So $\mathbb{H}(n)=\infty$ for $n>1$.

Corollary 1.2 is also proved in section 3 .

## 2. The averaging theory

Now we summarize the basic results from averaging theory that we need for proving the results of this paper. The following result provides a first order approximation for the periodic solutions of a periodic differential equation.

We deal with the differential equation

$$
\begin{equation*}
\dot{\mathbf{x}}=\varepsilon F_{1}(t, \mathbf{x})+\varepsilon^{2} F_{2}(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0)=\mathbf{x}_{0} \tag{3}
\end{equation*}
$$

with $\mathbf{x} \in D$, where $D$ is an open subset of $\mathbb{R}^{n}, t \geq 0$. Suppose that the functions $F_{1}(t, \mathbf{x})$ and $F_{2}(t, \mathbf{x}, \varepsilon)$ are $T$-periodic in $t$. Then consider in $D$ the averaged differential equation

$$
\begin{equation*}
\dot{\mathbf{y}}=\varepsilon f(\mathbf{y}), \quad \mathbf{y}(0)=\mathbf{x}_{0}, \tag{4}
\end{equation*}
$$

where

$$
f(\mathbf{y})=\frac{1}{T} \int_{0}^{T} F_{1}(t, \mathbf{y}) d t
$$

The next result shows that under convenient conditions, the equilibrium solutions of the averaged equation correspond with $T$-periodic solutions of the differential equation (3).

Theorem 2.1. Consider the two differential equations (3) and (4). Assume:
(i) the functions $F_{1}$, its Jacobian $\partial F_{1} / \partial x$, its Hessian $\partial^{2} F_{1} / \partial x^{2}$, $F_{2}$ and its Jacobian $\partial F_{2} / \partial x$ are continuous and bounded by a constant independent of $\varepsilon$ in the sets $[0, \infty) \times D$ and $\varepsilon \in\left(0, \varepsilon_{0}\right]$.
(ii) the functions $F_{1}$ and $F_{2}$ are $T$-periodic in $t$ ( $T$ independent of $\varepsilon)$.
Then the next statements hold.
(a) If $p$ is an equilibrium point of the averaged equation (4) and

$$
\left.\operatorname{det}\left(\frac{\partial f}{\partial \mathbf{y}}\right)\right|_{\mathbf{y}=p} \neq 0
$$

then there is a $T$-periodic solution $\varphi(t, \varepsilon)$ of equation (3) such that $\varphi(0, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.
(b) The kind of stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the kind of stability or instability of the equilibrium point $p$ of the averaged system (4). Indeed, the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.

For a proof of Theorem 2.1 see Theorems 11.5 and 11.6 of Verhulst [8].

## 3. Proof of Theorem 1.1

Consider the subclass of differential equations (2) with $n=1$ given by

$$
\begin{equation*}
\frac{d r}{d \theta}=\varepsilon \frac{a(\theta)}{a_{0}(\theta)+a_{1}(\theta) r}, \tag{5}
\end{equation*}
$$

where $\varepsilon$ is a small parameter, and

$$
\begin{equation*}
a(\theta)=\sum_{j=0}^{k} \alpha_{j} \cos (j \theta), \quad a_{0}(\theta)=1, \quad \text { and } a_{1}(\theta)=\cos \theta \tag{6}
\end{equation*}
$$

being $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{k}$ arbitrary constants.
Clearly the differential equation (5) is defined in the open cylinder $\left\{(\theta, r) \in \mathbb{S}^{1} \times(0,1)\right\}$. This differential equation satisfies the assumptions of Theorem 2.1, so we shall apply this theorem to it.
The averaged differential equation (4) corresponding to equation (5) is

$$
\begin{equation*}
\dot{r}=\varepsilon f(r) \tag{7}
\end{equation*}
$$

where

$$
f(r)=\sum_{j=0}^{k} \alpha_{j} \frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{\cos (j \theta)}{1+r \cos \theta} d \theta=\sum_{j=0}^{k} \alpha_{j} f_{j}(r)
$$

The function $f_{j}(r)$ for $r \in(0,1)$ can be computed, and we get

$$
\begin{equation*}
f_{j}(r)=\frac{1}{\sqrt{1-r^{2}}}\left(\frac{\sqrt{1-r^{2}}-1}{r}\right)^{j} \tag{8}
\end{equation*}
$$

In fact this integral was computed in the formula 3.613 of [2]. Therefore

$$
f(r)=\sum_{j=0}^{k} \alpha_{j} f_{j}(r)=\sum_{j=0}^{k} \alpha_{j} \frac{1}{\sqrt{1-r^{2}}}\left(\frac{\sqrt{1-r^{2}}-1}{r}\right)^{j}
$$

The equilibrium points of the averaged equation (7) are the zeros of the function $f(r)$.

Let $I$ be an interval of $\mathbb{R}$, and let $f_{0}, f_{1}, \ldots, f_{k}: I \rightarrow \mathbb{R}$ be $\mathcal{C}^{1}$ functions linearly independent, i.e. if $\sum_{j=0}^{k} \beta_{j} f_{j}(r)=0$ then $\beta_{0}=\beta_{1}=$ $\ldots=\beta_{k}=0$. The following result is well known, for a proof see for instance the Proposition 1 of the Appendix A of [7].

Proposition 3.1. If the functions $f_{0}, f_{1}, \ldots, f_{k}: I \rightarrow \mathbb{R}$ are linearly independent, then there exist $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{k} \in \mathbb{R}$ and $r_{1}, \ldots, r_{k} \in I$ such that for every $r_{\ell}$ with $\ell \in\{1, \ldots, k\}$ we have that

$$
\sum_{j=0}^{k} \alpha_{j} f_{j}\left(r_{\ell}\right)=0
$$

Clearly our functions $f_{j}(r)$ for $j=0,1, \ldots, k$ given in (8) are linearly independent. So we can apply Proposition 3.1 to them, and consequently we know that there are values of $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{k} \in \mathbb{R}$ and values $r_{1}, \ldots, r_{k} \in(0,1)$ such that $f\left(r_{\ell}\right)=0$ for $\ell=1, \ldots, k$, being the $r_{\ell}$ simple zeros of $f(r)$. Hence, by Theorem 2.1, since the averaged equation (7) has $k$ simple zeros $r_{1}, \ldots, r_{k} \in(0,1)$ we conclude that the differential equation (5) has $k$ limit cycles. This completes the proof of Theorem 1.1.

## 4. Proof of Corollary 1.2

We consider for a given integer $n>1$ the differential equation

$$
\begin{align*}
\frac{d r}{d \theta} & =\varepsilon \frac{a(\theta)}{a_{0}(\theta)+a_{1}(\theta) r+\varepsilon\left(a_{2}(\theta) r^{2}+\ldots+a_{n}(\theta) r^{n}\right)},  \tag{9}\\
& =\varepsilon \frac{a(\theta)}{a_{0}(\theta)+a_{1}(\theta) r}+\mathcal{O}\left(\varepsilon^{2}\right) .
\end{align*}
$$

Taking again the expressions (6) for the functions $a(\theta), a_{0}(\theta)$ and $a_{1}(\theta)$, we can apply Theorem 2.1 to the differential equation (9) as we have done for the differential equation (5), and we also obtain that the differential equation (9) has $k$ limit cycles. This completes the proof of the Corollary 1.2 .

## Acknowledgments

The first author is partially supported by a MINECO/FEDER grant MTM2008-03437, an AGAUR grant 2009SGR-0410, an ICREA Academia, FP7-PEOPLE-2012-IRSES-316338 and 318999, and FEDER/UNAB10-4E-378.

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[^0]:    Key words and phrases. Periodic orbit, averaging theory, trigonometric polynomial, Hilbert number.

