BIFURCATION OF LIMIT CYCLES FROM SOME UNIFORM ISOCHRONOUS CENTERS

JAUME LLIBRE[1] AND AMAR MAKHLOUF[2]

Abstract. This article concerns with the weak 16–th Hilbert problem. More precisely, we consider the uniform isochronous centers

$$\dot{x} = -y + x^{n-1}y, \qquad \dot{y} = x + x^{n-2}y^2,$$

for $n = 2, 3, 4$, and we perturb them by all homogeneous polynomial of degree $2, 3, 4$, respectively. Using averaging theory of first order we prove that the maximum number $N(n)$ of limit cycles that can bifurcate from the periodic orbits of the centers for $n = 2, 3$, under the mentioned perturbations, is 2. We prove that $N(4) \geq 2$, but there is numerical evidence that $N(4) = 2$. Finally we conjecture that using averaging theory of first order $N(n) = 2$ for all $n > 1$. Some computations have been made with the help of an algebraic manipulator as mathematica.

1. Introduction and statement of the main results

The second part of the 16th Hilbert's problem asks for the maximum number $H(n)$ and position of limit cycles for all planar polynomial differential systems of degree n, for more details on the 16th Hilbert's problem see [8, 10, 11], and the references quoted therein. The problem on the number $H(n)$ remains open, even for $n = 2$, and a general result about the configurations of limit cycles in planar polynomial differential systems can be found in [13].

A weak form of the 16th Hilbert's problem, known now as the *weak 16th Hilbert's problem* was proposed by Arnold [1], asking for the maximum number $Z(m, n)$ of isolated zeros of Abelian integrals of all polynomial 1–form of degree n over algebraic ovals of degree m, for more information on the weak 16th Hilbert's problem see [5, 9, 18], and the hundreds of references quoted therein. Of course $Z(m, n) \leq H(n)$. But the weak 16th Hilbert's problem is also extremely hard to study. In fact the weak 16th Hilbert's problem is the problem of studying the maximum number of limit cycles that can bifurcate from the periodic solutions of a polynomial differential center having a rational first integral of degree m when it is perturbed inside the class of all polynomial differential systems of degree n.

2010 *Mathematics Subject Classification.* 37G15, 37C80, 37C30.

Key words and phrases. periodic solution, uniform isochronous centers, averaging theory, weak Hilbert problem.

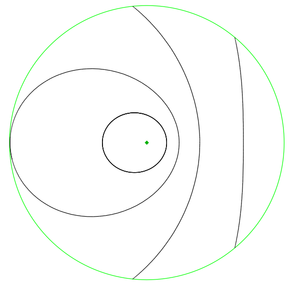

FIGURE 1. Phase portrait of the uniform isochronous center (1) with $n = 2$.

Our goal in this article will be to provide lower bounds for the maximum number of limit cycles that can bifurcate from the periodic solutions of a polynomial differential uniform isochronous center of degree $2, 3$ and 4 when it its perturbed by all homogeneous polynomial of degree $2, 3$ and 4, respectively. More precisely, we consider the polynomial differential system

$$\begin{aligned} \dot{x} &= -y + x^{n-1}y, \\ \dot{y} &= x + x^{n-2}y^2, \end{aligned} \tag{1}$$

of degree $n \geq 2$, having a uniform isochronous center at the origin of coordinates, which in polar coordinates (r, θ), where $x = r\sin\theta$ and $y = r\cos\theta$, becomes

$$\begin{aligned} \dot{r} &= r^n \cos^{n-2}\theta \sin\theta, \\ \dot{\theta} &= 1. \end{aligned}$$

As usual the dot in the previous two differential systems denotes derivative with respect to an independent variable t, usually called the time. Since $\dot{\theta} = 1$ the center (1) is uniform and isochronous, which taking as independent variable the variable θ writes

$$\frac{dr}{d\theta} = r' = r^n \cos^{n-2}\theta \sin\theta. \tag{2}$$

An easy computation shows that the periodic solutions $r(\theta, r_0)$ surrounding the center $r = 0$ such that $r(0, r_0) = r_0$ are

$$r(\theta, r_0) = r_0 \left(1 - r_0^{n-1} + r_0^{n-1}\cos^{n-1}\theta\right)^{\frac{1}{1-n}}, \tag{3}$$

with $0 < r_0 < 1$ if n is odd, and $0 < r_0 < 2^{1/(1-n)}$ if n is even. System (1) has the rational first integral

$$H = \frac{(1 - x^{n-1})^2}{(x^2 + y^2)^{n-1}}.$$

So the periodic solutions of the center (1) are algebraic ovals of degree $2(n-1)$. Hence when we perturb the center (1) by all homogeneous polynomial of degree n we are studying a particular case of the weak Hilbert problem.

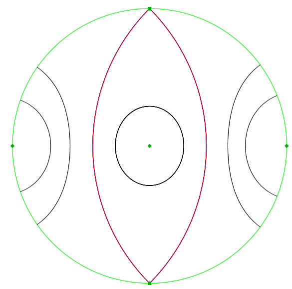

FIGURE 2. Phase portrait of the uniform isochronous center (1) with $n = 3$.

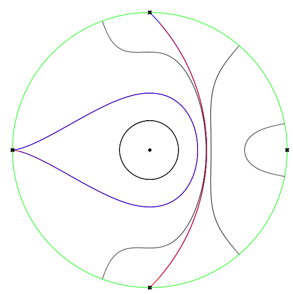

FIGURE 3. Phase portrait of the uniform isochronous center (1) with $n = 4$.

The phase portraits of the uniform isochronous centers (1) for $n = 2, 3, 4$ in the Poincaré disc are given in Figures 1, 2, 3, respectively. For more details on the Poincaré disc see Chapitre 5 of [7].

Our goal is to provide a lower bound for the maximum number of limit cycles that can bifurcate from the periodic solutions $r(0, r_0) = r_0$ surrounding the uniform isochronous center at $r = 0$ of degree $n = 2, 3, 4$ when we perturb by all homogeneous polynomials of degree $2, 3, 4$, respectively. More precisely, we want to study the maximum number of limit cycles of the following three polynomial differential systems

$$\begin{aligned} \dot{x} &= -y + xy + \varepsilon(a_0x^2 + a_1xy + a_2y^2), \\ \dot{y} &= x + y^2 + \varepsilon(b_0x^2 + b_1xy + b_2y^2); \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{x} &= -y + x^2y + \varepsilon(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3), \\ \dot{y} &= x + xy^2 + \varepsilon(b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3); \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{x} &= -y + x^3y + \varepsilon(a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4), \\ \dot{y} &= x + x^2y^2 + \varepsilon(b_0x^4 + b_1x^3y + b_2x^2y^2 + b_3xy^3 + a_4y^4); \end{aligned} \tag{6}$$

where ε is a small parameter.

Our main result is the following.

Theorem 1. *For $|\varepsilon| \neq 0$ sufficiently small and using averaging of first order the following statements hold.*

- (a) *Systems* (4) *have at most* 2 *limit cycles bifurcating from the periodic orbits of the center* (1) *with $n = 2$, and there are systems* (4) *with* 2 *limit cycles.*
- (b) *Systems* (5) *have at most* 2 *limit cycles bifurcating from the periodic orbits of the center* (1) *with $n = 3$, and there are systems* (5) *with* 2 *limit cycles.*
- (c) *There are systems* (6) *with at least* 2 *limit cycles bifurcating from the periodic orbits of the center* (1) *with $n = 4$.*

The proof of statements (a), (b) and (c) of Theorem 1 are done in sections 2, 3 and 4 respectively.

We have numerical evidence that the best statement (c) of Theorem 1 would be: *Systems* (6) *have at most* 2 *limit cycles bifurcating from the periodic orbits of the center* (1) *with $n = 4$, and there are systems* (6) *with* 2 *limit cycles.*

In fact in the plane $\mathbb{R}^2$ the averaging theory of first order, or the generalized Abelian integrals, or the Melnikov function provide the same information because they are based in the first term in ε of the Poincaré return map.

Chicone and Jacobs in [4] proved that perturbing any quadratic polynomial isochronous center inside the class of all quadratic polynomial differential systems at most 2 limit cycles can bifurcate. Our result for systems (4) shows that already these 2 limit cycles appear when we perturbed the quadratic uniform isochronous center (1) by the restricted class of homogeneous quadratic polynomials.

Using averaging theory of first order it follows from statements (a) and (b) of Theorem 1 that the maximum number $N(n)$ of limit cycles of systems (4) and (5) that can bifurcate from the periodic orbits of the centers (1) for $n = 2$ and $n = 3$, respectively, is 2. And from statement (c) of Theorem 1 we know that $N(4) \geq 2$, but as we said there is numerical evidence that $N(4) = 2$. Consequently we do the following conjecture.

Conjecture. *Using averaging theory of first order the maximum number $N(n)$ of limit cycles of the system*

$$\begin{aligned}
\dot{x} &= -y + x^{n-1}y + \varepsilon \sum_{k=0}^{n} a_k x^{n-k} y^k, \\
\dot{y} &= x + x^{n-2}y^2 + \varepsilon \sum_{k=0}^{n} b_k x^{n-k} y^k;
\end{aligned}$$

that can bifurcate from the periodic orbits of the center (1) *is* 2.

For recent studies on isochronous and uniform isochronous centers see for instance [12] and [15].

2. Proof of statement (a) of Theorem 1

In polar coordinates (r,θ), where $x = r\sin\theta$ and $y = r\cos\theta$, system (4) becomes

$$\begin{aligned}\dot r = \;& r^2\sin\theta + \varepsilon r^2(a_0\cos^3\theta + (a_1+b_0)\cos^2\theta\sin\theta + \\ & (a_2+b_1)\cos\theta\sin^2\theta + b_2\sin^3\theta),\\ \dot\theta = \;& 1 + \varepsilon r(b_0\cos^3\theta - (a_0-b_1)\cos^2\theta\sin\theta - \\ & (a_1-b_2)\cos\theta\sin^2\theta - a_2\sin^3\theta).\end{aligned}$$

Taking as independent variable the variable θ, we obtain the equivalent differential equation

$$\begin{aligned} r' = \;& r^2\sin\theta + \varepsilon\Big(r^2\big(a_0\cos^3\theta + (a_1+b_0)\cos^2\theta\sin\theta \\ & +(a_2+b_1)\cos\theta\sin^2\theta + b_2\sin^3\theta\big) + r^3\big(-b_0\cos^3\theta\sin\theta \\ & +(a_0-b_1)\cos^2\theta\sin^2\theta + (a_1-b_2)\cos\theta\sin^3\theta + a_2\sin^4\theta\big)\Big) \\ & +O(\varepsilon^2) \\ = \;& F_0(\theta,r) + \varepsilon F_1(\theta,r) + O(\varepsilon^2).\end{aligned} \tag{7}$$

Here the prime denotes derivative with respect to the variable θ. This differential equation is in the normal form (12) for applying the averaging theory described in the Appendix.

The periodic solution $r(\theta,r_0)$ given in (3) of the unperturbed equation (2) for $n=2$ is

$$r(\theta,r_0) = \frac{r_0}{1-r_0+r_0\cos\theta}.$$

The variational differential equation of equation (7) at this periodic solution is

$$\frac{dM}{d\theta} = \frac{2r_0\sin\theta}{1-r_0+r_0\cos\theta}M.$$

Its solution such that $M(0)=1$ is

$$M(\theta) = \frac{1}{(1-r_0+r_0\cos\theta)^2}.$$

Now we must compute the averaged function (15) of the appendix, which in our case is

$$I(r_0) = \int_0^{2\pi} \frac{1}{M(\theta)} F_1(\theta, r(\theta,r_0))d\theta. \tag{8}$$

An easy computation shows that

$$
\begin{aligned}
I(r_0) = & \int_0^{2\pi} \Big(\frac{a_0(r_0-1)r_0^2\cos^3\theta}{1-r_0+r_0\cos\theta} + \frac{a_0 r_0^3\cos^4\theta}{1-r_0+r_0\cos\theta} \\
& - \frac{(a_2+b_1)(r_0-1)r_0^2\cos\theta\sin^2\theta}{1-r_0+r_0\cos\theta} + \frac{(a_0+a_2)r_0^3\cos^2\theta\sin^2\theta}{1-r_0+r_0\cos\theta} \\
& + \frac{a_2 r_0^3\sin^4\theta}{1-r_0+r_0\cos\theta}\Big) d\theta \\
= & \frac{\pi(2a_2\sqrt{1-2r_0}-(a_0+a_2-b_1))r_0^3}{2-2\sqrt{1-2r_0}(r_0-1)-4r_0+r_0^2}.
\end{aligned}
$$

The zeros of $I(r_0)=0$ are

$$
0, \quad 1 + \frac{2a_2}{-2a_2 \pm \sqrt{a_0+3a_2-b_1}\sqrt{-a_0+a_2+b_1}}.
$$

Consequently at most there are two positive zeros, and it is easy to have examples with two zeros, so by Theorem 3 of the Appendix we get at most two limit cycles. This completes the proof of statement (a) of Theorem 1.

3. Proof of statement (*b*) of Theorem 1

In polar coordinates (r,θ) system (5) becomes

$$
\begin{aligned}
\dot r = & \; r^3\cos\theta\sin\theta + \varepsilon r^3(a_0\cos^4\theta + (a_1+b_0)\cos^3\theta\sin\theta + \\
& (a_2+b_1)\cos^2\theta\sin^2\theta + (a_3+b_2)\cos\theta\sin^3\theta + b_3\sin^4\theta), \\
\dot\theta = & \; 1 + \varepsilon r^2(b_0\cos^4\theta - (a_0-b_1)\cos^3\theta\sin\theta - \\
& (a_1-b_2)\cos^2\theta\sin^2\theta - (a_2-b_3)\cos\theta\sin^3\theta - a_3\sin^4\theta).
\end{aligned}
$$

Taking as independent variable the variable θ we obtain the equivalent differential equation

$$
\begin{aligned}
r' = & \; r^3\cos\theta\sin\theta + \varepsilon r^3\Big(a_0\cos^4\theta + (a_1+b_0)\cos^3\theta\sin\theta \\
& + (a_2+b_1)\cos^2\theta\sin^2\theta + (a_3+b_2)\cos\theta\sin^3\theta + b_3\sin^4\theta \\
& + r^2\cos\theta\sin\theta\big(-b_0\cos^4\theta + (a_0-b_1)\cos^3\theta\sin\theta \\
& + (a_1-b_2)\cos^2\theta\sin^2\theta + (a_2-b_3)\cos\theta\sin^3\theta + a_3\sin^4\theta\big)\Big) \\
& + O(\varepsilon^2) \\
= & \; F_0(\theta,r) + \varepsilon F_1(\theta,r) + O(\varepsilon^2).
\end{aligned} \tag{9}
$$

Again this differential equation is written in the normal form (12) for applying the averaging theory of the Appendix.

The periodic solutions $r(\theta,r_0)$ given in (3) for the unperturbed equation (2) when $n=3$ are

$$
r(\theta,r_0) = \frac{r_0}{(1-r_0^2+r_0^2\cos^2\theta)^{\frac{1}{2}}}.
$$

The variational differential equation of equation (9) at this periodic solution is

$$\frac{dM}{d\theta} = \frac{3r_0^2 \sin\theta\cos\theta}{1 - r_0^2 + r_0^2\cos^2\theta} M.$$

Its solution such that $M(0) = 1$ is

$$M(\theta) = \frac{1}{(1 - r_0^2 + r_0^2\cos^2\theta)^{\frac{3}{2}}}.$$

Now we must compute the averaged function (15) of the appendix, which in our case is an in (8). An easy computation shows that

$$\begin{aligned}
I(r_0) = \;& \int_0^{2\pi} \left(\frac{r_0^3\left(b_1 r_0^2 - b_3 r_0^2 + a_0 - a_2 - b_1 + b_3\right)\cos^4\theta}{8\left(1 - r_0^2 + r_0^2\cos^2\theta\right)} \right.\\
& - \frac{3r_0^3\left(b_1 r_0^2 - b_3 r_0^2 + a_0 - a_2 - b_1 + b_3\right)\sin^2\theta\cos^2\theta}{4\left(1 - r_0^2 + r_0^2\cos^2\theta\right)}\\
& + \frac{r_0^3\left(b_3 r_0^2 + a_0 - b_3\right)\cos^2\theta}{2(1 - r_0^2 + r_0^2\cos^2\theta)}\\
& + \frac{r_0^3\left(-b_1 r_0^2 - 3b_3 r_0^2 + 3a_0 + a_2 + b_1 + 3b_3\right)}{8(1 - r_0^2 + r_0^2\cos^2\theta)}\\
& + \frac{r_0^3\left(b_1 r_0^2 - b_3 r_0^2 + a_0 - a_2 - b_1 + b_3\right)\sin^4\theta}{8(1 - r_0^2 + r_0^2\cos^2\theta)}\\
& \left. - \frac{r_0^3\left(b_3 r_0^2 + a_0 - b_3\right)\sin^2\theta}{2(1 - r_0^2 + r_0^2\cos^2\theta)} \right) d\theta\\
= \;& \frac{\pi}{r_0}\Big(\big(b_1(r_0^2 - 1) - a_2\big)(r_0^2 - 2 + 2\sqrt{1 - r_0^2}) + b_3\big(r_0^4 + r_0^2 - 2 +\\
& 2\sqrt{1 - r_0^2}\big) + a_0\big(r_0^2(3 - 2\sqrt{1 - r_0^2}) + 2(-1 + \sqrt{1 - r_0^2})\big)\Big).
\end{aligned}$$

The unique two possible positive zeros of $I(r_0) = 0$ are

$$\sqrt{1 - \frac{2(a_0 + b_3)^2}{(b_1 + b_3)^2} + \frac{a_0 + a_2}{b_1 + b_3} \pm \frac{2\sqrt{(a_0 + b_3^2)^2(a_0^2 + b_3^2 + a_0(b_3 - b_1) - a_2(b_1 + b_3))}}{(b_1 + b_3)^2}}.$$

So, by Theorem 3 of the Appendix, we get at most two limit cycles and it is easy produce examples with two. This completes the proof of statement (b) of Theorem 1.

4. Proof of statement (c) of Theorem 1

In polar coordinates (r,θ) system (5) becomes

$$\begin{aligned}
\dot r = &\ \cos^2\theta\sin\theta r^4+\varepsilon\Big(a_0\cos^5\theta+(a_1+b_0)\sin\theta\cos^4\theta\\
&+(a_2+b_1)\sin^2\theta\cos^3\theta+(a_3+b_2)\sin^3\theta\cos^2\theta+(a_4+b_3)\sin^4\theta\cos\theta\\
&+b_4\sin^5\theta\Big)r^4,\\
\dot\theta = &\ 1+\varepsilon\Big(b_0\cos^5\theta-(a_0-b_1)\sin\theta\cos^4\theta-(a_1-b_2)\sin^2\theta\cos^3\theta\\
&-(a_2-b_3)\sin^3\theta\cos^2\theta-(a_3-b_4)\sin^4\theta\cos\theta-a_4\sin^5\theta\Big)r^3.
\end{aligned}$$

Taking as independent variable the variable θ we obtain the equivalent differential equation

$$\begin{aligned}
r' = &\ r^4\cos^2\theta\sin\theta+\varepsilon\, r^4\Big(a_0\cos\theta^5+(a_1+b_0)\cos\theta^4\sin\theta\\
&+(a_2+b_1)\cos\theta^3\sin\theta^2+(a_3+b_2)\cos\theta^2\sin\theta^3\\
&+(a_4+b_3)\cos\theta\sin\theta^4+b_4\sin\theta^5\Big)+\varepsilon\, r^7\Big(-b_0\cos\theta^7\sin\theta\\
&+(a_0-b_1)\cos\theta^6\sin\theta^2+(a_1-b_2)\cos\theta^5\sin\theta^3\\
&+(a_2-b_3)\cos\theta^4\sin\theta^4+(a_3-b_4)\cos\theta^3\sin\theta^5\\
&+a_4\cos\theta^2\sin\theta^6\Big)+O(\varepsilon^2)\\
= &\ F_0(\theta,r)+\varepsilon F_1(\theta,r)+O(\varepsilon^2).
\end{aligned}\tag{10}$$

This differential equation is written in the normal form (12) for applying the averaging theory. The periodic solutions $r(\theta,r_0)$ given in (3) for the unperturbed equation (2) when $n=4$ are

$$r(\theta,r_0)=\frac{r_0}{(1-r_0^3+r_0^3\cos^3\theta)^{\frac{1}{3}}}.$$

The variational differential equation of equation (10) at this periodic solution is

$$\frac{dM}{d\theta}=\frac{4r_0^3\cos^2\theta\sin\theta}{1-r_0^3+r_0^3\cos^3\theta}M.$$

Its solution such that $M(0)=1$ is

$$M(\theta)=\frac{1}{(1-r_0^3+r_0^3\cos^3\theta)^{\frac{4}{3}}}.$$

Now we must compute the averaged function (15) of the appendix, which in our case is an in (8). After some easy computations we have

$$
\begin{aligned}
I(r_0) = \;& 2\int_0^\pi \Bigg(-\frac{(a_0-a_2+a_4-b_1+b_3)r_0^{10}\cos^{11}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&+\frac{(a_0-2a_2+3a_4-b_1+2b_3)r_0^{10}\cos^{9}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&+\frac{(a_0-a_2+a_4-b_1+b_3)r_0^{7}\cos^{8}\theta}{1-r_0^3+r_0^3\cos^3\theta} \\
&+\frac{(a_0-a_2+a_4-b_1+b_3)(r_0-1)r_0^{7}\left(r_0^2+r_0+1\right)\cos^{8}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&+\frac{(a_2-3a_4-b_3)r_0^{10}\cos^{7}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&+\frac{(a_2-2a_4+b_1-2b_3)r_0^{7}\cos^{6}\theta}{1-r_0^3+r_0^3\cos^3\theta} \\
&-\frac{(a_0-2a_2+3a_4-b_1+2b_3)(r_0-1)r_0^{7}\left(r_0^2+r_0+1\right)\cos^{6}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&-\frac{(a_0-a_2+a_4-b_1+b_3)(r_0-1)r_0^{4}\left(r_0^2+r_0+1\right)\cos^{5}\theta}{1-r_0^3+r_0^3\cos^3\theta} \\
&+\frac{a_4r_0^{10}\cos^{5}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&+\frac{(a_4+b_3)r_0^{7}\cos^{4}\theta}{1-r_0^3+r_0^3\cos^3\theta} \\
&-\frac{(a_2-3a_4-b_3)(r_0-1)r_0^{7}\left(r_0^2+r_0+1\right)\cos^{4}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&-\frac{(a_2-2a_4+b_1-2b_3)(r_0-1)r_0^{4}\left(r_0^2+r_0+1\right)\cos^{3}\theta}{1-r_0^3+r_0^3\cos^3\theta} \\
&-\frac{a_4(r_0-1)r_0^{7}\left(r_0^2+r_0+1\right)\cos^{2}\theta}{\left(1-r_0^3+r_0^3\cos^3\theta\right)^2} \\
&-\frac{(a_4+b_3)(r_0-1)r_0^{4}\left(r_0^2+r_0+1\right)\cos\theta}{1-r_0^3+r_0^3\cos^3\theta}\Bigg)\,d\theta \\
= \;& \frac{1}{6}r_0(f(\pi,r_0)-f(0,r_0)),
\end{aligned}
$$

where $r_0 \in$ and

$$f(\theta, r_0) = 6(a_0 - 3a_2 + 5a_4 - b_1 + 3b_3)(r_0^3 - 1)\theta + 4\sum_{\lambda_k} g(\theta, \lambda_k, r_0),$$

here λ_k run over the six roots of the polynomial

$$p(\lambda, r_0) = \left(2r_0^3 - 1\right)\lambda^6 - 3\lambda^4 + 3\left(2r_0^3 - 1\right)\lambda^2 - 1,$$

and

$$\begin{aligned}
g(\theta, \lambda, r_0) = \;& \frac{g_1(\theta, \lambda, r_0)}{g_2(\theta, \lambda, r_0)},\\
g_1(\theta, \lambda, r_0) = \;& \Big(\big(2(a_2 - 3a_4 - b_3)\lambda^4 + 2(3a_2 - 2a_0 - 3a_4 + 2b_1 - 3b_3)\lambda^2\big)r_0^6\\
& +\big((7a_4 - 3a_2 + 3b_3)\lambda^4 + 4(2a_0 - 3a_2 + 4a_4 - 2b_1 + 3b_3)\lambda^2\\
& -a_2 + a_4 + b_3\big)r_0^3 + (a_2 - 2a_4 - b_3)\lambda^4 + a_2 - 2a_4 - b_3\\
& +(6a_2 - 4a_0 - 8a_4 + 4b_1 - 6b_3)\lambda^2\Big)\log\left(\tan(\theta/2) - \lambda\right),\\
g_2(\theta, \lambda, r_0) = \;& (2r_0^3 - 1)(\lambda^5 - \lambda) - 2\lambda^3.
\end{aligned}$$

The function $f(\theta, r_0)$ has been computed with the help of mathematica. We note that

$$f(\pi, r_0) = \lim_{\theta \uparrow \pi} f(\theta, r_0),$$

and that this limit exists because the area of the functions that appear in the integral $I(r_0)$ is finite.

Now it is easy to check using an algebraic manipulator as mathematica that

$$I(r_0) = (a_0 + b_1)f_0(r_0) + (a_2 + b_3)f_1(r_0) + a_4 f_2(r_0), \tag{11}$$

and that the functions $f_i(r_0)$ for $i = 0, 1, 2$ are linearly independent. The following result is well–known for a proof see for instance [14].

Proposition 2. *Let $f_0, \ldots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. If $f_0, \ldots, f_n$ are linearly independent then there exists $s_1, \ldots, s_n \in I$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, \ldots, n\}$ we have that s_j is a simple zero of the function $\displaystyle\sum_{i=0}^{n} \lambda_i f_i(s)$.*

Applying Proposition 2 to the function $I(r_0)$ given in (11) it follows that $I(r_0)$ has 2 simple zeros. Therefore, by Theorem 3 statement (c) of Theorem 1 is proved.

Appendix: Basic results on averaging theory

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\mathbf{x}' = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{12}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\mathbf{x}' = F_0(t,\mathbf{x}), \tag{13}$$

has a submanifold of dimension n of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},0)$ be the solution of the system (13) such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{14}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (14).

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (12); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 3 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t,\mathbf{x}(t,\mathbf{z},0))dt. \tag{15}$$

If there exists $\alpha \in V$ with $\mathcal{F}(\alpha) = 0$ and $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(\alpha)\right) \neq 0$, then there exists a T–periodic solution $\varphi(t,\varepsilon)$ of system (12) such that $\varphi(0,\varepsilon) \to \alpha$ as $\varepsilon \to 0$.

Theorem 3 is due to Malkin [16] and Roseau [17], for a shorter proof see [2].

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014 SGR568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, FEDER-UNAB-10-4E-378.

References

[1] V.I. Arnold, *Ten problems*, Adv. Soviet Math. **1** (1990), 1–8.

[2] A. Buică, J. P. Françoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Comm. on Pure and Appl. Anal. **6** (2007), 103–111.

[3] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qualitative Theory of Dynamical Systems **1** (1999), 1–70.

[4] C. Chicone and M. Jacobs, *Bifurcation of limit cycles from quadratic isochronous*, J. Differential Equations **91** (1991), 268–326.

[5] C. Christopher and C. Li, *Limit cycles in differential equations*, Birkhäuser, Boston, 2007.

[6] R. Conti, *Uniformly isochronous centers of polynomial systems in* $\mathbb{R}^2$, Lecture Notes in Pure and Appl. Math. **152** (1994), 21–31.

[7] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Spring-Verlag, 2006.

[8] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. Göttingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479.

[9] V. Kaloshin, *Around the Hilbert-Arnol'd problem. On finiteness in differential equations and Diophantine geometry*, CRM Monograph Series 24, Edited by D. Schlomiuk, Amer. Math. Soc. Providence, RI, 2005, pp 111–162.

[10] Yu. Ilyashenko, *Centennial history of Hilbert's* 16*th problem*, Bull. (New Series) Amer. Math. Soc. **39** (2002), 301–354.

[11] J. Li, *Hilbert's* 16*th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.

[12] C. Liu and M. Han, *Bifurcation of critical periods from the reversible rigidly isochronous centers* Nonlinear Anal. **95** (2014), 388–403.

[13] J. Llibre and G. Rodríguez, *Configuration of limit cycles and planar polynomial vector fields*, J. Differential Equations **198** (2004), 374–380.

[14] J. Llibre and G. Świrszcz, *On the limit cycles of polynomial vector fields*, Dynamics of Continuous, Discrete and Impulsive Systems, Serie A **18** (2011), 203–214.

[15] J. Llibre and C. Valls, *Classification of the centers and isochronous centers for a class of quartic-like systems*, Nonlinear Anal. **71** (2009), 3119–3128.

[16] I.G. Malkin, *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956.

[17] M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol.**8** Springer–Verlag, Berlin–New York, 1966.

[18] S. Yakovenko, *Qualitative theory of ordinary differential equations and tangential Hilbert 16th problem*, CRM Monograph Series 24, Edited by D. Schlomiuk, Amer. Math. Soc. Providence, RI, 2005, pp 41–109.

[1] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[2] Department of Mathematics, University of Annaba, Elhadjar, 23 Annaba, Algeria

E-mail address: `makhloufamar@yahoo.fr`