PERIODIC SOLUTIONS OF SOME CLASSES OF CONTINUOUS SECOND–ORDER DIFFERENTIAL EQUATIONS

JAUME LLIBRE[1] AND AMAR MAKHLOUF[2]

Abstract. We study the periodic solutions of the second–order differential equations of the form $\ddot{x} \pm x^n = \mu f(t)$, or $\ddot{x} \pm |x|^n = \mu f(t)$, where $n = 4, 5, \ldots$, $f(t)$ is a continuous T–periodic function such that $\int_0^T f(t)dt \neq 0$, and μ is a positive small parameter. Note that the differential equations $\ddot{x} \pm x^n = \mu f(t)$ are only continuous in t and smooth in x, and that the differential equations $\ddot{x} \pm |x|^n = \mu f(t)$ are only continuous in t and locally–Lipschitz in x.

1. Introduction and statement of the main results

The periodic solutions of the second–order differential equations

$$\ddot{x} + x^3 = f(t), \tag{1}$$

where $f(t)$ is a T–periodic function have been studied by several authors. Thus, Morris [6] proves that if $f(t)$ is C^1 and its averaged is zero (i.e. $\int_0^T f(t)dt = 0$), then the differential equation (1) has periodic solutions of period kT for all positive integer k. Ding and Zanolin [4] proved the same result without the assumption that the averaged of $f(t)$ be zero. Almost there is no results on the stability of these periodic solutions, but Ortega [7] proved that the differential equation (1) has finitely many stable periodic solutions of a fixed period.

Our goal is to extend the mentioned results on the periodic solutions of the second–order differential equation (1) to the second–order differential equations of the form

$$\ddot{x} \pm x^n = \mu f(t), \tag{2}$$

and

$$\ddot{x} \pm |x|^n = \mu f(t), \tag{3}$$

2010 *Mathematics Subject Classification.* 37G15, 37C80, 37C30.
Key words and phrases. periodic solution, second order differential equations, averaging theory.

where $n = 4, 5, \ldots$, $f(t)$ is a continuous T–periodic function such that $\int_0^T f(t)dt \neq 0$, and $\mu > 0$ is a small parameter. Moreover, we shall study the linear stability or instability of such periodic solutions.

Note that the differential equations (2) are only *continuous* in t and smooth in x, and that the differential equations (3) are only *continuous* in t and *locally–Lipschitz* in x. As far as we know these kind of differential equations have not been studied up to know.

Our main results are the following two theorems.

Theorem 1. *Consider the second–order differential equations*

$$\ddot{x} \pm x^n = \mu f(t), \tag{4}$$

where $n = 4, 5, \ldots$, $f(t)$ is continuous, T–periodic function such that $\int_0^T f(t)dt \neq 0$, and $\mu > 0$ is a small parameter. Then, for $\mu > 0$ sufficiently small there exist two periodic solutions $x_\pm(t, \mu)$ of period T of the differential equation (4) such that

$$x_\pm(0, \mu) = \pm\mu^{1/n} \left| \pm\frac{1}{T}\int_0^T f(t)dt \right|^{1/n} + O(\mu^{(n-1)/(2n)}), \tag{5}$$

if either $\pm \int_0^T f(t)dt > 0$ when n is even, or when n is odd. Moreover the periodic solution $x_-(t, \mu)$ is unstable for the equation $\ddot{x} + x^n = \mu f(t)$ if n is even, and for the equations $\ddot{x} \pm x^n = \mu f(t)$ if n is odd.

Theorem 1 is proved in section 2.

Note that we are using in (5) and in the rest of the paper the following notation: for the solutions

$$x_+(0, \mu) = \mu^{1/n} \left(+\frac{1}{T}\int_0^T f(t)dt \right)^{1/n} + O(\mu^{(n-1)/(2n)}), \tag{6}$$

and

$$x_-(0, \mu) = \mu^{1/n} \left(-\frac{1}{T}\int_0^T f(t)dt \right)^{1/n} + O(\mu^{(n-1)/(2n)}), \tag{7}$$

we only write (5).

Theorem 2. *Consider the second–order differential equations*

$$\ddot{x} \pm |x|^n = \mu f(t), \tag{8}$$

where $n = 4, 5, \ldots$, $f(t)$ is continuous, T–periodic function such that $\int_0^T f(t)dt \neq 0$, and $\mu > 0$ is a small parameter. Then, for μ sufficiently small there exist two periodic solutions $x_\pm(t,\mu)$ of period T of the differential equation (8) *such that*

$$x_\pm(0,\mu) = \pm\mu^{1/n}\left|\frac{1}{T}\int_0^T f(t)dt\right|^{1/n} + O(\mu^{(n-1)/(2n)}), \tag{9}$$

if either $\pm\int_0^T f(t)dt > 0$ when n is even, or when n is odd. Moreover, the periodic solutions $x_\pm(t,\mu)$ for the equation $\ddot{x} - |x|^n = \mu f(t)$ are unstable.

Let $g : \mathbb{R} \to \mathbb{R}$ be the 2-periodic function defined by

$$g(t) = \begin{cases} t & if \quad t \in [0,1], \\ 2-t & if \quad t \in [1,2]. \end{cases}$$

The following two corollaries follow easily from the previous two theorems.

Corollary 3. *For $\mu > 0$ sufficiently small the equations $\ddot{x} \pm x^4 = \mu g(t)$ have two periodic solutions $x_\pm(t,\mu)$ such that $x(0,\mu) = \pm\sqrt[4]{\mu/2} + O(\mu^{3/8})$.*

Corollary 4. *For μ sufficiently small then equations $\ddot{x} + |x|^4 = \mu\sin^2 t$ have two periodic solutions $x_\pm(t,\mu)$ such that $x_\pm(0,\mu) = \pm\sqrt[4]{\mu/2} + O(\mu^{3/8})$.*

2. Proof of the results

In this section we shall prove Theorems 1 and 2, and Corollaries 3 and 4.

Proof of Theorem 1. Under the assumptions of Theorem 1 we write the second–order differential equation as the differential system of first order

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mp x^n + \mu f(t). \end{aligned} \tag{10}$$

Doing the change of variables

$$x = \varepsilon^{2/(n-1)}X, \quad y = \varepsilon^{(n+1)/(n-1)}Y, \quad \mu = \varepsilon^{(2n)/(n-1)}, \tag{11}$$

with $\varepsilon > 0$, the differential system (10) becomes

$$\begin{aligned} \dot{X} &= \varepsilon Y, \\ \dot{Y} &= \varepsilon\big(\mp X^n + f(t)\big). \end{aligned} \tag{12}$$

We note that the change of variables (11) is well defined because $n > 1$. Now we apply the averaging theory of first order of the appendix. Using the notation of Theorem 5 of the appendix system (12) can be written as system (15) with $\mathbf{x} = (X, Y)$, $H = (Y, \mp X^n + f(t))$, $R = (0, 0)$. The averaged function $h(\mathbf{z})$ given in (16) for system (12) becomes

$$h(X, Y) = \left(Y, \mp X^n + \frac{1}{T} \int_0^T f(t) dt \right).$$

If n is even then the function $h(X, Y)$ has two unique zeros

$$(X_\pm^*, X_\pm^*) = (\pm(\pm\frac{1}{T} \int_0^T f(t) dt)^{1/n}, 0).$$

when $\pm\frac{1}{T} \int_0^T f(t) dt > 0$ for the equation $\ddot{x} \pm x^n = \mu f(t)$; note that only one of these two differential equations has two periodic solutions. If n is odd then the function $h(X, Y)$ has two zeros,

$$(X_\pm^*, Y_\pm^*) = ((\pm\frac{1}{T} \int_0^T f(t) dt)^{1/n}, 0),$$

when $\int_0^T f(t) dt \neq 0$ for both equations $\ddot{x} \pm x^n = \mu f(t)$.

The Jacobian of the function h(X,Y) at theses zeros is $\pm n X_\pm^{*(n-1)}$. By Theorem 5 and Remark 1 we deduce that there are two periodic solutions $(X_\pm(t, \varepsilon), Y_\pm(t, \varepsilon))$ of system (12) satisfying that

$$(X_\pm(0, \varepsilon), Y_\pm(0, \varepsilon)) = \left(X_\pm^*, 0\right) + O(\varepsilon).$$

From (11) we have $x = \mu^{1/n} X$. We conclude that for $\mu > 0$ sufficiently small there exist two periodic solutions $x_\pm(t, \mu)$ of period T of the differential equation (4) such that

$$x_\pm(0, \mu) = \mu^{1/n} X_\pm^* + O(\mu^{(n-1)/(2n)}).$$

We note that for $\mu > 0$ sufficiently small $\mu^{1/n} \gg \mu^{(n-1)/(2n)}$ if and only if $n > 3$, which holds by assumption.

The two eigenvalues of the corresponding Jacobian matrix of the averaged function $h(X, Y)$ at the zero (X^*, Y^*) are $\pm\sqrt{\mp n X_\pm^{*(n-1)}}$. If n is even and $\pm\frac{1}{T} \int_0^T f(t) dt > 0$ the solution $(X_-(t, \varepsilon), Y_-(t, \varepsilon))$ of system (12) provides an unstable periodic solution for the equation $\ddot{x} + x^n = \mu f(t)$. If n is odd and $\frac{1}{T} \int_0^T f(t) dt \neq 0$ the solution

$(X_-(t,\varepsilon), Y_-(t,\varepsilon))$ of system (12) provides an unstable periodic solution for the equation $\ddot{x} \pm x^n = \mu f(t)$. Then from Theorem 6 of this appendix it follows the results on the instability of the periodic solutions stated in the theorem. □

Proof of Theorem 2. In the assumptions of Theorem 2 we write the second–order differential equation as the differential system of first order

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \mp |x|^n + \mu f(t).
\end{aligned} \tag{13}
$$

Doing the change of variables (11), the differential system (13) becomes

$$
\begin{aligned}
\dot{X} &= \varepsilon Y, \\
\dot{Y} &= \varepsilon\big(\mp |X|^n + f(t)\big).
\end{aligned} \tag{14}
$$

Note that we can apply the averaging theory of first order of the appendix because the function $|X|^n$ is locally Lipschitz. Using the notation of Theorem 5 of the appendix system (14) can be written as system (15) with $\mathbf{x} = (X, Y)$, $H = (Y, \mp|X|^n + f(t))$, $R = (0, 0)$. The averaged function $h(\mathbf{z})$ given in (16) for system (14) becomes

$$
h(X,Y) = \left(Y, \mp |X|^n + \frac{1}{T}\int_0^T f(t)dt \right).
$$

The function $h(X, Y)$ has the two zeros

$$
\big(X^*_\pm, Y^*_\pm\big) = \left(\pm \left(\pm \frac{1}{T}\int_0^T f(t)dt \right)^{1/n}, 0 \right),
$$

such zeros exist when $\pm \displaystyle\int_0^T f(t)dt > 0$ and n is even, or when $\displaystyle\int_0^T f(t)dt \neq 0$ and n is odd. The Jacobians of the function h(X,Y) at the zeros $\big(X^*_\pm, Y^*_\pm\big)$ are $\pm n|X^*_\pm|^{n-1}$. By Theorem 5 and Remark 1 we deduce that there is two periodic solutions $(X_\pm(t,\varepsilon), Y_\pm(t,\varepsilon))$ of system (14) satisfying that

$$
(X_\pm(0,\varepsilon), Y_\pm(0,\varepsilon)) = \big(X^*_\pm, 0\big) + O(\varepsilon).
$$

Since $x = \varepsilon^{2/(n-1)}X$ and $\mu = \varepsilon^{(2n)/(n-1)}$, we have $x = \mu^{1/n}X$. So for $\mu > 0$ sufficiently small there exists two periodic solutions $x_\pm(t,\mu)$ of period T of the differential equation (13) such that

$$
x_\pm(0,\mu) = \mu^{1/n} X^*_\pm + O(\mu^{(n-1)/(2n)}).
$$

The two eigenvalues of the corresponding Jacobian matrix of the averaged function $h(X, Y)$ at the zeros $(X^*_\pm, 0)$ are $\pm\sqrt{-n|X^*_\pm|^{n-1}}$ for the equation $\ddot{x} + |x|^n = \mu f(t)$, and at the zeros $(X^*_\pm, 0)$ are $\pm\sqrt{n|X^*_\pm|^{n-1}}$

for the equation $\ddot{x} - |x|^n = \mu f(t)$. Again by Theorem 6 it follows that the periodic solutions $x_\pm(t,\mu)$ are unstable for the equation $\ddot{x} - |x|^n = \mu f(t)$. This completes the proof of the theorem. □

Appendix: averaging theory of first order

In this section we present the first order averaging method as it was extended in [1], where the differentiability of the vector field is not needed. The sufficient conditions for the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree, see [5] for precise definitions.

Theorem 5. *We consider the following differential system*

$$\dot{\mathbf{x}}(t) = \varepsilon H(t,\mathbf{x}) + \varepsilon^2 R(t,\mathbf{x},\varepsilon), \tag{15}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*–periodic in* t*, and* D *is an open subset of* $\mathbb{R}^n$*. We define* $h : D \to \mathbb{R}^n$ *as*

$$h(\mathbf{z}) = \frac{1}{T}\int_0^T H(s,\mathbf{z})ds, \tag{16}$$

and assume that

- (i) H *and* R *are locally Lipschitz in* x*;*
- (ii) *for* $\mathbf{a} \in D$ *with* $h(\mathbf{a}) = 0$*, there exists a neighborhood* V *of* $\mathbf{a}$ *such that* $h(\mathbf{z}) \neq 0$ *for all* $\mathbf{z} \in \overline{V}\setminus\{\mathbf{a}\}$ *and* $d_B(h,V,\mathbf{a}) \neq 0$ *(where* $d_B(h,V,\mathbf{a})$ *denotes the Brouwer degree of* h *in the neighborhood* V *of* $\mathbf{a}$*).*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists an isolated* T*–periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system* (15) *such that* $\mathbf{x}(0,\varepsilon) \to \mathbf{a}$ *as* $\varepsilon \to 0$*.*

If the averaged function $h(\mathbf{z})$ is differentiable in some neighborhood of a fixed isolated zero $\mathbf{a}$ of $h(\mathbf{z})$, then we can use the following remark in order to verify the hypothesis *(ii)* of Theorem 5. For more details see again [5].

Remark 1. *Let* $h : D \to \mathbb{R}^n$ *be a* C^1 *function, with* $h(\mathbf{a}) = 0$*, where* D *is an open subset of* $\mathbb{R}^n$ *and* $\mathbf{a} \in D$*. Whenever* $\mathbf{a}$ *is a simple zero of* h $(\det(Dh(\mathbf{a})) \neq 0)$*, i.e the determinant of the Jacobian matrix of the function* h *at* $\mathbf{a}$ *is not zero), there exists a neighborhood* V *of* $\mathbf{a}$ *such that* $h(\mathbf{z}) \neq 0$ *for all* $\mathbf{z} \in \overline{V} \setminus \{\mathbf{a}\}$*. Then* $d_B(h,\mathbf{a},V,0) \in \{-1,1\}$*.*

In [2] Theorem 5 is improved as follows.

Theorem 6. *Under the assumptions of Theorem 5, for small* ε *the condition* $\det(Dh(\mathbf{a})) \neq 0$ *ensures the existence and uniqueness of a*

$T-$periodic solution $\mathbf{x}(t,\varepsilon)$ of system (15) such that $\mathbf{x}(0,\varepsilon) \to \mathbf{a}$ as $\varepsilon \to 0$, and if all eigenvalues of the matrix $Dh(\mathbf{a})$ have negative real parts, then the periodic solution $\mathbf{x}(t,\varepsilon)$ is stable. If some of the eigenvalue has positive real part the periodic solution $\mathbf{x}(t,\varepsilon)$ is unstable.

The averaging theory for studying periodic solutions is very useful see for instance [3].

Acknowledgements

We thank to Professor Rafael Ortega the information about the second–order differential equation $\ddot{x} + x^3 = f(t)$.

The first author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014 SGR568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

References

[1] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** 2004, 7–22.

[2] A. Buică, J. Llibre and O.Yu. Makarenkov, *On Yu.A.Mitropol'skii's Theorem on periodic solutions of systems of nonlinear differential equations with nondifferentiable right-hand sides* Doklady Math.,**78** (2008), 525-527.

[3] T. Carvalho, R.D. Euzébio, J. Llibre and D.J. Tonon, *Detecting periodic orbits in some 3D chaotic quadratic polynomial differential systems*, Discrete and Continuous Dynamical Systems- Series B **21** (2016), 1–11.

[4] T.R. Ding and F. Zanolin, *Periodic solutions of Duffing's equations with superquadratic potential*, J. Differential Equations **97** (1992), 328–378.

[5] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.

[6] G.R. Morris, *An infinite class of periodic solutions of $\ddot{x} + 2x^3 = p(t)$*, Proc. Cambridge Philos. Soc. **61** (1965), 157–164.

[7] R. Ortega, *The number of stable periodic solutions of time–dependent Hamiltonian systems with one degree of freedom*, Ergodic Theory Dynam. Systems **18** (1998), 1007–1018.

[1] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[2] Department of Mathematics, Laboratory LMA, University of Annaba, Elhadjar, 23 Annaba, Algeria

E-mail address: makhloufamar@yahoo.fr