Research Article

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Periodic solutions for periodic second-order differential equations with variable potentials

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Abstract: We provide sufficient conditions for the existence of periodic solutions of the second-order differential equation with variable potentials $-(px')'(t) - r(t)p(t)x'(t) + q(t)x(t) = f(t, x(t))$, where the functions $p(t) > 0$, $q(t)$, $r(t)$ and $f(t, x)$ are $\mathcal{C}^2$ and T-periodic in the variable t.

Keywords: Periodic orbit, third-order differential equation, quadratic system, averaging theory

MSC 2010: 37G15, 37C80, 37C30

1 Introduction and statement of the main result

We want to study the periodic solutions of the second-order differential equation with variable potentials given by

$$-(px')'(t) - r(t)p(t)x'(t) + q(t)x(t) = f(t, x(t)), \tag{1.1}$$

where the functions $p(t) > 0$, $q(t)$, $r(t)$ and $f(t, x)$ are T-periodic. Here the prime denotes derivative with respect to the time t.

The T-periodic differential equation (1.1) has been considered by several authors. Liu, Ge and Gui [6] (see also [2]) studied it with $r(t) = 0$. Graef, Kong and Wang [5] give an extensive analysis when the functions $p(t)$, $q(t)$ and $r(t)$ are constant. More recently, Anderson and Avery [3] also studied the periodic solutions of the differential equation (1.1) with $p(t) > 0$, $q(t) > 0$ and $r(t) \geq 0$.

Here we study the periodic solutions of the differential equation (1.1) with the unique basic assumption that the functions $p(t) > 0$, $q(t)$, $r(t)$ and $f(t, x)$ are $\mathcal{C}^2$ and T-periodic in the variable t.

Instead of working with the T-periodic second-order differential equation (1.1), we shall work with the following equivalent T-periodic differential system of first order:

$$\begin{aligned} x' &= y, \\ y' &= \frac{q(t)}{p(t)}x - \Big(r(t) + \frac{p'(t)}{p(t)}\Big)y - \frac{f(t, x)}{p(t)}. \end{aligned} \tag{1.2}$$

Our results on the periodic solutions of the differential system (1.2) are summarized in the next theorem.

Theorem 1. *We consider the differential system* (1.2), *where the functions* $p(t) > 0$, $q(t)$, $r(t)$ *and* $f(t, x)$ *are* $\mathcal{C}^2$ *and* T*-periodic in the variable* t*. Then the following statements hold.*

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