DYNAMICS OF A FAMILY OF LOTKA–VOLTERRA SYSTEMS IN $\mathbb{R}^3$

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Abstract. We provide the phase portraits of the 3–dimensional Lotka–Volterra systems

$$\dot{x} = x(y + az), \quad \dot{y} = y(x + z), \quad \dot{z} = bz(-ax + y),$$

for all the values of the parameters a and b.

1. Introduction and statement of the main results

We say that a polynomial vector field $X = (P(x, y, z), Q(x, y, z), R(x, y, z))$ in $\mathbb{R}^3$ is quadratic if the maximum of the degrees of the polynomials P, Q and R is 2. A quadratic polynomial vector field X with x a factor of P, y a factor of Q, and z a factor of R is by definition a Lotka–Volterra system.

The Lotka–Volterra systems, which are quadratic polynomial differential systems of degree 2, were initially proposed in $\mathbb{R}^2$ independently by Alfred J. Lotka in 1925 [16] and by Vito Volterra in 1926 [23], as a model for studying the interactions between species. Later on Kolmogorov [12] in 1936 extended these systems to arbitrary dimension and arbitrary degree, these kinds of systems are now called Kolmogorov systems.

Many natural phenomena can be modeled by the Lotka–Volterra systems such as the time evolution of conflicting species in biology [18], chemical reactions [9], hydrodynamics [6], economics [22], the coupling of waves in laser physics [13], the evolution of electrons, ions and neutral species in plasma physics [14], etc. After the work of Brenig and Goriely [4, 5] the interest in the Lotka–Volterra systems becomes more important, because they proved that many other differential systems can be transformed into 3-dimensional Lotka–Volterra systems by using a quasimonomial formalism.

In general the dynamics of the Lotka–Volterra systems are far from being understood, although some dynamics for special families of these systems have been revealed (see [1], [3], [15], [24], [25]). Thus for instance, the theory on cooperative or competitive systems was developed by Hirsch in the papers [10]-[11], where he proved that these systems generically exhibits a global attractor which lies on a 2-dimensional manifold.

In this work we consider the following class of Lotka–Volterra systems

$$\begin{aligned} \dot{x} &= x(y + az), \\ \dot{y} &= y(x + z), \\ \dot{z} &= bz(-ax + y), \end{aligned} \tag{1}$$

which depends on the two parameters a and b. Here the dot denotes derivative with respect to the time t.

The phase portrait of a 3-dimensional differential system is determined completely if we know two first integrals whose gradients are linearly independent in $\mathbb{R}^3$ except perhaps in a zero Lebesgue measure set. This is due to the fact that the trajectories of the system are determined by intersections of the invariant levels of these two first integrals. But if we know a unique first integral then the study of the dynamics of the 3–dimensional differential system can be reduced to study a 2–dimensional differential system. So an important subject in the

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qualitative theory of differential systems is the study of the existence of first integrals. In the next, we establish the existence of first integrals for system (1).

Note that system (1) has the first integral

$$H(x,y,z) = b(x-y) + z,$$

because on the solutions $(x(t), y(t), z(t))$ of system (1) we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} = 0.$$

We can reduce the study of the dynamics of the Lotka–Volterra system (1) in $\mathbb{R}^3$ to study its dynamics on the planes $H(x,y,z) = h$, where h varies in $\mathbb{R}$. More precisely, at the energy level $H(x,y,z) = h$ we have $z = b(y-x) + h$, so the analysis of the differential system (1) is reduced to analyze the systems

$$\begin{aligned} \dot{x} &= x(ah - abx + (1+ab)y), \\ \dot{y} &= y(h + (1-b)x + by), \end{aligned} \tag{2}$$

for all $h \in \mathbb{R}$. Note that system (2) depends on the three parameters a, b and h.

Proposition 1. *System* (2) *with* $b \neq 0$ *on the energy level* $H(x,y,z) = h$ *is integrable with the first integral* $H_2(x,y,z) = -x^{-b}y^{ab}(h - bx + by)$.

Proof. It is easily to check that on the solutions $(x(t), y(t))$ of the system (2) we have

$$\frac{dH_2}{dt} = \frac{\partial H_2}{\partial x}\dot{x} + \frac{\partial H_2}{\partial y}\dot{y} = 0.$$

□

Note that system (2) when $b = 0$ takes the form

$$\begin{aligned} \dot{x} &= x(y + ah), \\ \dot{y} &= y(x + h). \end{aligned} \tag{3}$$

Proposition 2. *System* (2) *with* $b = 0$ *on the energy level* $H(x,y,z) = h$ *is integrable with the first integral* $H_3(x,y,z) = e^{x-y}x^h y^{-ah}$.

Proof. Again it is easily to check that on the solutions $(x(t), y(t))$ of the system (3) we have

$$\frac{dH_3}{dt} = \frac{\partial H_3}{\partial x}\dot{x} + \frac{\partial H_3}{\partial y}\dot{y} = 0.$$

□

From Propositions 1 and 2 it follows the next result.

Corollary 3. *The Lotka–Volterra systems* (1) *in* $\mathbb{R}^3$ *are completely integrable with first integrals* $H_1 = H(x,y,z)$ *and* H_2 *if* $b \neq 0$, *and with the integrals* H_1 *and* H_3 *if* $b = 0$.

We recall that a *limit cycle* for a Lotka–Volterra system (1) is a periodic orbit which is isolated in the set of all periodic orbits of the system.

Corollary 4. *The Lotka–Volterra systems* (1) *have no limit cycles.*

Proof. If a Lotka–Volterra system (1) has a limit cycle, then some differential system (2) has a limit cycle, but Bautin in [2] proved that any 2–dimensional Lotka–Volterra system cannot have limit cycles. □

The objective of this work is to describe the dynamics of the Lotka–Volterra systems (1) in $\mathbb{R}^3$ adding the infinity. Clearly, from Propositions 1 and 2 this is equivalent to describe the phase portraits on the Poincaré disc of the 2–dimensional Lotka–Volterra systems (2) for the different values of its three parameters.

Roughly speaking the Poincaré disc $\mathbb{D}^2$ is the closed unit disc centered at the origin of $\mathbb{R}^2$, where its interior is identified with $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ is defined as the infinity $\mathbb{R}^2$, in the sense that in the plane $\mathbb{R}^2$ we can go to or come from the infinity in as many directions as points has the circle $\mathbb{S}^1$. A polynomial differential system in $\mathbb{R}^2$, i.e. in the interior of $\mathbb{D}^2$ can be extended to the its boundary $\mathbb{S}^1$ in a unique analytic way, this extension is called the Poincaré compactification of a polynomial differential system, because was done by first time by Poincaré in [21].

We say that two compactified polynomial differential systems on the Poincaré disc $\mathbb{D}^2$ are *topologically equivalent* if there is a homeomorphism of $\mathbb{D}^2$ which send orbits of one system into orbits of the other system preserving or reversing the orientation of all the orbits.

Theorem 5. *The phase portrait in the Poincaré disc of a 2–dimensional Lotka–Volterra system (2) on the surface $z = b(y - x) + h$, where $H(x, y, z) = h \in \mathbb{R}$, is topologically equivalent to one of the 18 phase portraits of Figure 1. Two bifurcation diagrams are presented in Figure 2(a) for $h \neq 0$ and in Figure 2(b) for $h = 0$.*

This work is organized as follows. In section 2 we present some basic definitions and some preliminary results necessary to prove Theorem 5. In section 3 we prove Theorem 5.

2. Preliminaries

In order to give a detailed proof of Theorem 5 we give some definitions and results that will be useful.

2.1. Phase portraits in the Poincaré disc.

Now we shall describe the equations of the Poincaré compactification for a polynomial differential system in $\mathbb{R}^2$. We consider the polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

or equivalently its associated polynomial vector field $\mathcal{X} = (P, Q)$. The degree n of $\mathcal{X}$ is defined as $n = \max\{\deg(P^i) : i = 1, 2\}$.

In the disc $\mathbb{D}^2$ we consider the local charts (U_k, ϕ_k) and (V_k, ψ_k) for $k = 1, 2$ defined as follows

$$U_k = \{x = (x_1, x_2) \in \mathbb{D}^2 : x_k > 0\}, \quad V_k = \{x = (x_1, x_2) \in \mathbb{D}^2 : x_k < 0\},$$

the $\phi_k : U_k \to \mathbb{R}^3$ for $k = 1, 2$ are

$$\phi_1(x) = \left(\frac{x_2}{x_1}, \frac{1}{x_1}\right) = (z_1, z_2), \quad \phi_2(x) = \left(\frac{x_1}{x_2}, \frac{1}{x_2}\right) = (z_1, z_2),$$

and $\psi_k(x) = -\phi_k(x)$.

Note that the coordinates (z_1, z_2) have different meaning in each local chart, but the points of the infinity, i.e. the points of the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$ all have the coordinate $z_2 = 0$.

The expression of the compactified analytical vector field $p(\mathcal{X})$ of the polynomial vector field $\mathcal{X}$ of degree n on the local chart U_1 of $\mathbb{D}^2$ is

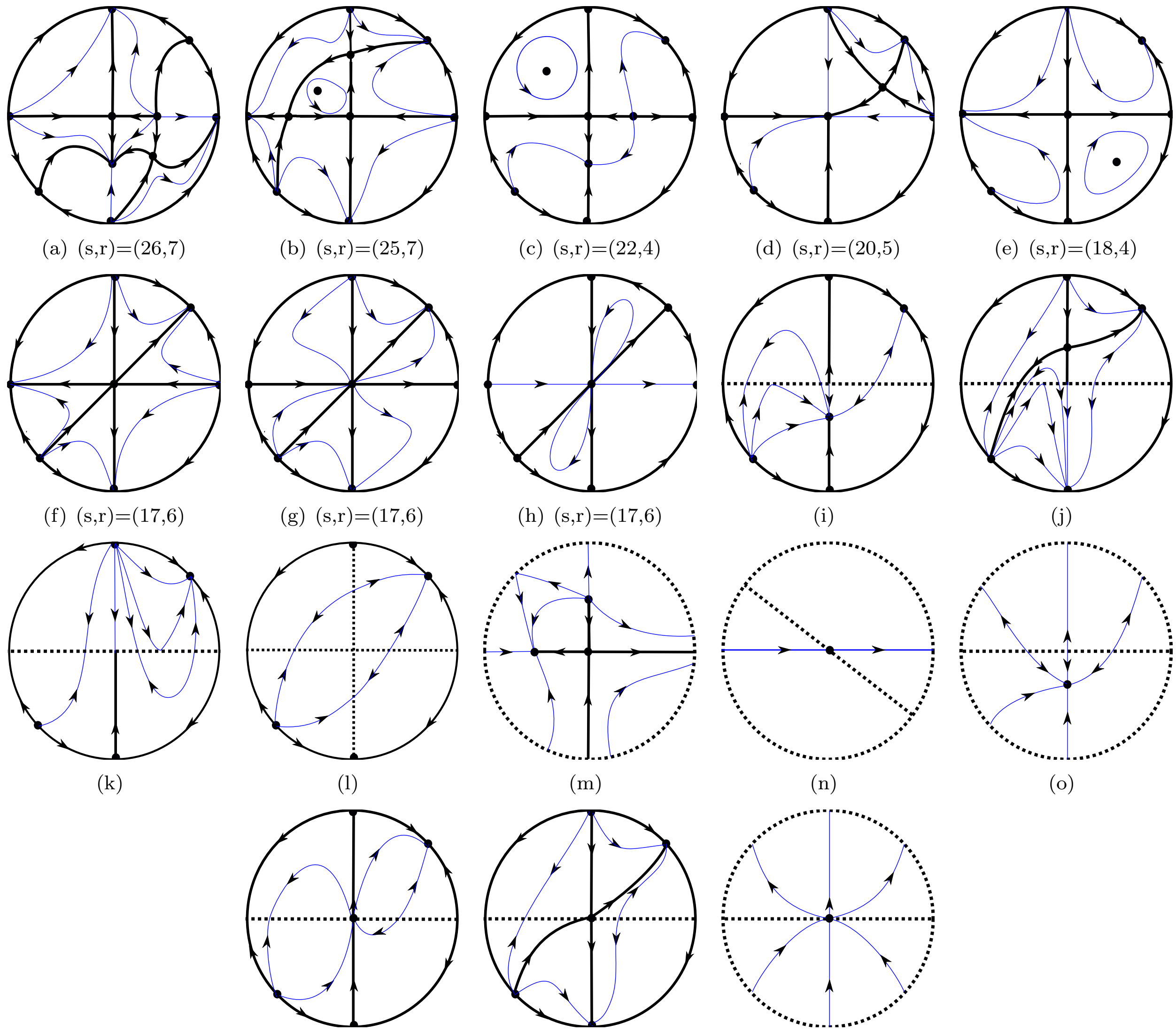
$$z_2^n \left(-z_1 P + Q, -z_2 P\right), \tag{4}$$

where $P^i = P^i\left(1/z_2, z_2/z_1\right)$.

In a similar way the expression of $p(\mathcal{X})$ in U_2 is

$$z_2^n \left(-z_1 Q + P, -z_2 Q\right), \tag{5}$$

where $P^i = P^i\left(z_1/z_2, 1/z_2\right)$.



(a) (s,r)=(26,7) (b) (s,r)=(25,7) (c) (s,r)=(22,4) (d) (s,r)=(20,5) (e) (s,r)=(18,4)

(f) (s,r)=(17,6) (g) (s,r)=(17,6) (h) (s,r)=(17,6) (i) (j)

(k) (l) (m) (n) (o)

(p) (q) (r)

FIGURE 1. Phase portrait of system (2). (a): $h \neq 0 \wedge ((a < 0 \wedge b > 1/(1-a)) \vee (0 < a < 1 \wedge (b < 0 \vee 0 < b < 1/(1-a))) \vee (a = 1 \wedge b \neq 1/(1-a)) \vee (a > 1 \wedge (0 < b < 1/(1-a) \vee b > 1/(1-a))))$; (b): $h \neq 0 \wedge a < 0 \wedge b < 0$, (c): $h \neq 0 \wedge ((a < 0 \wedge 0 < b < 1/(1-a)) \vee (0 < a < 1 \wedge b > 1/(1-a)) \vee (a > 1 \wedge b < 0))$, (d): $h \neq 0 \wedge ((0 < a < 1 \wedge b = 0) \vee (a \geq 1 \wedge b = 1/(1-a)))$, (e): $h \neq 0 \wedge b = 0$, (f): $h = 0 \wedge a < 0 \wedge b < 0$, (g): $h = 0 \wedge ((a < 0 \wedge b > 1/(1-a)) \vee (0 < a < 1 \wedge b < 1/(1-a)) \vee (a = 1 \wedge b \neq 1/(1-a)) \vee (a > 1 \wedge (0 < b < 1/(1-a) \vee b > 1/(1-a))))$, (h): $h = 0 \wedge (((a < 0) \wedge 0 < b < 1/(1-a)) \vee (0 < a < 1 \wedge (0 < b < 1/(1-a) \vee b > 1/1-a))) \vee (a > 1 \wedge b < 0))$, (i): $h \neq 0 \wedge a = 0 \wedge (0 < b < 1/(1-a) \vee b > 1/(1-a))$, (j): $h \neq 0 \wedge a = 0 \wedge b < 0$, (k): $h \neq 0 \wedge a = 0 \wedge b = 0$, (l): $h = 0 \wedge ((0 \leq a < 1 \wedge b = 0) \vee (a \geq 1 \wedge b = 1/(1-a)))$, (m): $h \neq 0 \wedge ((a < 0 \wedge b = 1/(1-a)) \vee (0 < a < 1 \wedge b = 1/(1-a)) \vee (a > 1 \wedge b = 0))$, (n): $h = 0 \wedge ((a < 0 \wedge b = 1/(1-a)) \vee (0 < a < 1 \wedge b = 1/(1-a)) \vee (a > 1 \wedge b = 0))$, (o): $h \neq 0 \wedge a = 0 \wedge b = 1$, (p): $h = 0 \wedge a = 0 \wedge (0 < b < 1/(1-a) \vee b > 1/(1-a))$, (q): $h = 0 \wedge a = 0 \wedge b < 0$, (r): $h = 0 \wedge a = 0 \wedge b = 1$.

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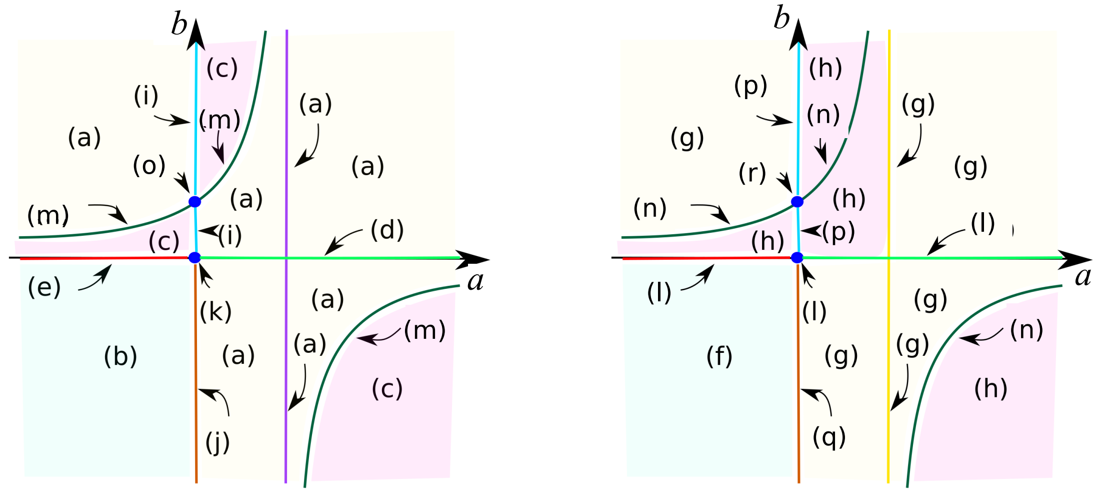


(a) (b)

FIGURE 2. Bifurcation diagrams of the phase portrait system (1) shown in Figure 1: (a) when $h \neq 0$, and (b) when $h = 0$.

The singular points of $p(\mathcal{X})$ which are on the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$ are called *infinite singular points*, and the ones which are in the interior of $\mathbb{D}^2$ are called *finite singular points*.

From (4) and (5) it follows that the infinity $\mathbb{S}^1$ of the Poincaré disc is invariant under the flow of the compactified vector field $p(\mathcal{X})$, and that for studying its infinite singular points we only need to study the ones on the local chart U_1 and the origin of the local chart U_2 in case that this be a singular point.

The expression for $p(\mathcal{X})$ in the local chart V_k is the same as in U_k multiplied by $(-1)^{n-1}$. Therefore the infinite singular points appear on pairs diametrally opposite on $\mathbb{S}^1$.

For more details on the Poincaré compactification see chapter 5 of [8].

Now we shall see how to characterize the phase portrait of a compactified vector field $p(X)$ in the Poincaré disc.

A *separatrix* of $p(X)$ being X a polynomial vector field defined in $\mathbb{R}^2$ is an orbit which is either an equilibrium point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite or an infinite equilibrium point, or any orbit contained at the infinity of the Poincaré disc, or a limit cycle. Neumann [19] proved that the set formed by all separatrices of $p(X)$, denoted by $S(p(X))$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(X))$ are called canonical regions of X or of $p(X)$. A *separatrix configuration* is the union of $S(p(X))$ plus one orbit chosen in each canonical region. Two separatrix configurations $S(p(X))$ and $S(p(\mathcal{Y}))$ are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(X))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [17], Neumann [19] and Peixoto [20], who found it independently.

Theorem 6. *The phase portraits in the Poincaré disc $\mathbb{D}^2$ of two compactified polynomial vector fields $p(X)$ and $p(\mathcal{Y})$ are topologically equivalent, if and only if, their separatrix configurations $S(p(X))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

2.2. **General results for planar homogeneous polynomial differential systems.** Let

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

be a a planar homogeneous polynomial differential system, and let $F(x, y) = xQ(x, y) - yP(x, y)$. To study the infinite singular points of $X = (P, Q)$ we consider the induced vector field $p(X)$ on the Poincaré disc.

The next result is proved in [7].

Proposition 7. *Let $X = (P, Q)$ be a homogeneous polynomial vector field in the plane with degree $(P) =$ degree$(Q) = n$ and assume that P and Q have no common factors. Assume that $F(x, y) = xQ(x, y) - yP(x, y)$ has some real linear factor. Then the following statements holds.*

(a) *The linear factor $ax + by$ of $F(x, y)$ provides the invariant straight line $ax + by = 0$ for the flow of X.*
(b) *X has no limit cycles.*
(c) *The singular points at infinity are all elementary and they are nodes, saddles, or saddles-nodes. An infinite singular point on the local chart U_1, $(z_1, z_2) = (\lambda_i, 0)$ shall be a saddle-node if and only if λ_i is a root of $f(\lambda) = F(1, \lambda) = Q(1, \lambda) - \lambda P(1, \lambda)$ of even multiplicity. Furthermore, the orbits in the Poincaré disc near a saddle-node are drawn in Fig. 3.*
(d) *The behavior of the flow of $p(X)$ in a neighborhood of infinity determines the phase portrait of X (Fig. 4 shows the possible behavior at infinity between two consecutive invariant rays of X).*

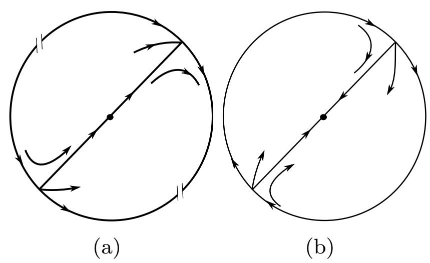


FIGURE 3. Behavior of the orbits of a compactified homogeneous polynomial differential system $p(X)$ near a saddle–node at infinity (we can reverse the orientation of the orbits): (a) when n even, and (b) when n odd.

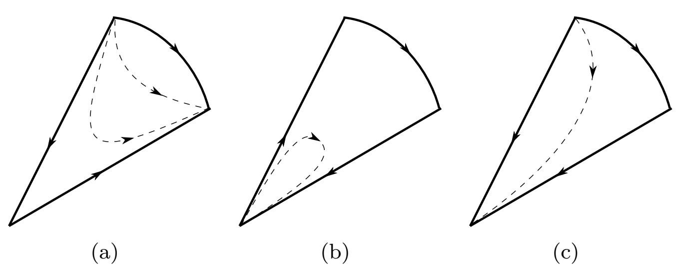


FIGURE 4. The behavior in a neighborhood of the infinity of a compactified homogeneous polynomial differential system determines its phase portrait.

3. Phase Portraits of system (2)

In this section we study all the possible phase portraits of the 2–dimensional Lotka–Volterra systems (2) in the Poincaré disc. Initially we study the finite and infinite singular points as function of the parameters a, b and h.

3.1. **Case** $ab(b-1/(1-a)) \neq 0$**.** First note that system (2) has always two invariant straight lines that are $x=0$ and $y=0$.

We start with the study of the infinite singular points, for this purpose we use the Poincaré compactification. We consider the system (2) in local chart U_1

$$\dot{z}_1 = z_1((1+(a-1)b)z_1 + (a-1)hz_2), \quad \dot{z}_2 = -z_2(-ab + (1+ab)z_1 + ahz_2). \tag{6}$$

At the infinity of the chart U_1, i.e. at $z_2 = 0$, system (6) has two singular points, the origin and $(1,0)$. The linear part of system (6) at $z_2 = 0$ is

$$\begin{pmatrix} -2(1+(a-1)b)\,z_1 + 1 + (a-1)b & -(a-1)h\,z_1 \\ 0 & ab - ab\,z_1 - z_1 \end{pmatrix}. \tag{7}$$

At the origin the matrix (7) is diagonal and their eigenvalues are $\lambda_1 = 1 + (a-1)b$ and $\lambda_2 = ab$, which are non–zero due to the condition $ab(b-1/(1-a)) \neq 0$. So the origin is a stable node if $a < 0$ and $b > 1/(1-a)$, or $a > 1$ and $b < 1/(1-a)$; it is an unstable node if $a < 0$ and $b < 0$, or $0 < a < 1$ and $0 < b < 1/(1-a)$, or $a \geq 1$ and $b > 0$; it is a saddle with local stable manifold on the z_2-axis if $a < 0$ and $0 < b < 1/(1-a)$, or $0 < a \leq 1$ and $b < 0$, or $a > 1$ and $1/(1-a) < b < 0$; and it is a saddle with local stable manifold on the z_1-axis if $0 < a < 1$ and $b > 1/(1-a)$.

On the other hand, the singular point $(1,0)$ has eigenvalues -1 and $-1 + b(1-a) \neq 0$. Then it is a stable node if $a < 1$ and $b < 1/(1-a)$, or $a > 1$ and $b > 1/(1-a)$, or if $a = 1$; and it is a saddle (with local stable manifold next to z_1-axis) if $a < 1$ and $b > 1/(1-a)$, or $a > 1$ and $b < 1/(1-a)$.

On the local chart U_2 system (2) becomes

$$\dot{z}_1 = -z_1(-1-(a-1)b + ((a-1)b+1)z_1 + h(1-a)z_2), \quad \dot{z}_2 = -z_2(b + (1-b)z_1 + hz_2). \tag{8}$$

It is clear that the origin of U_2 is a singular point, and its linear matrix is

$$\begin{pmatrix} 1 + b(a-1) & 0 \\ 0 & -b \end{pmatrix}. \tag{9}$$

Then its eigenvalues are $\lambda_1 = 1 + b(a-1)$ and $\lambda_2 = -b$, and consequently distinct from zero. Thus the origin of U_2 is an unstable node if $a \leq 1$ and $b < 0$, or $a > 1$ and $1/(1-a) < b < 0$; it is a stable node if $a < 1$ and $b > 1/(1-a)$; it is a saddle with local stable manifold on the z_2-axis if $a < 1$ and $0 < b < 1/(1-a)$, or $a \geq 1$ and $b > 0$; and it is a saddle with local stable manifold on the z_1-axis if $a > 1$ and $b < 1/(1-a)$.

In the next we study the finite singular points. For this purpose we separate the cases when $h = 0$ and $h \neq 0$, due to the fact that the number and type of singular points depends on this parameter.

3.1.1. *Subcase* $h \neq 0$. For $h \neq 0$ system (2) has four finite equilibria: namely

$$e_1 = (0,0), \quad e_2 = (0, -h/b), \quad e_3 = (h/b, 0) \text{ and } e_4 = \left(\frac{-h}{1+b(a-1)}, \frac{-ah}{1+b(a-1)}\right).$$

The linear matrix of system (2) is

$$\begin{pmatrix} ah - 2abx + (ab+1)y & (ab+1)x \\ (1-b)y & h + (1-b)x + 2by \end{pmatrix} \tag{10}$$

In Tables 1 and 2 are explicit the local phase portrait of the four finite singular points as function of the parameters a, b and h.

a	h	$b<0$	$0<b<1/(1-a)$	$b=1/(1-a)$	$b>1/(1-a)$
$a<0$	$h>0$	$S1-S2-S2-C$	$S1-SN-UN-C$	$S1-SN-UN-*$	$S1-SN-UN-S$
	$h<0$	$S2-S1-S1-C$	$S2-UN-SN-C$	$S2-UN-SN-*$	$S2-UN-SN-S$
$0<a<1$	$h>0$	$UN-S2-SN-S$	$UN-SN-S1-S$	$UN-SN-S1-*$	$UN-SN-S1-C$
	$h<0$	$SN-S1-UN-S$	$SN-UN-S2-S$	$SN-UN-S2-*$	$SN-UN-S2-C$
$a=1$	$h>0$	$UN-S2-SN-S$	$UN-SN-S1-S$	$UN-SN-S1-*$	$UN-SN-S1-S$
	$h<0$	$SN-S1-UN-S$	$SN-UN-S2-S$	$SN-UN-S2-*$	$SN-UN-S2-S$

TABLE 1. Local phase portraits at the finite singular points in the order e_1-e_2-e_3-e_4 for $a \leq 1$. UN: Unstable node, SN: stable node, C: center, S: saddle, S1: saddle with local stable manifold on x–axis, S2: saddle with local stable manifold on y–axis. (* represent that singular point e_4 is not define).

a	h	$b<1/(1-a)$	$b=1/(1-a)$	$1/(1-a)<b<0$	$b>0$
$a>1$	$h>0$	$UN-S2-SN-C$	$UN-S2-SN-*$	$UN-S2-SN-S$	$UN-SN-S1-S$
	$h<0$	$SN-S1-UN-C$	$SN-S1-UN-*$	$SN-S1-UN-S$	$SN-UN-S2-S$

TABLE 2. Local phase portraits at the finite singular points in the order e_1-e_2-e_3-e_4 for $a > 1$.

From these tables note that the local phase portraits of the four finite singular points does not depend on the sign of h. We consider three different combination of finite singular points (distinguished with different colors in the cells of the tables):

- (a) two saddles, one stable node and one unstable node if $a<0, b>1/(1-a)$, or $0<a<1, 0<b<1/(1-a)$, or $a=1$ or $a>1, b>1/(1-a)$.
- (b) three saddles and one center if $a<0$ and $b<0$.
- (c) one center, one saddle, one stable node and one unstable node if $a<0, 0<b<1/(1-a)$, or $0<a<1, b>1/(1-a)$, or $a>1, b<1/(1-a)$.

Even more, if in the finite region we have the mentioned singular points in (a), then the infinite singular points (in counterclockwise order) are a stable node, a saddle, a stable node, an unstable node, a saddle, and an unstable node. If the finite region is the described in (b), then the infinite singular points (in counterclockwise order) are an unstable node, a stable node, an unstable node, a stable node, an unstable node and a stable node. Finally, if the finite region is described by (c), then the infinite singular points (in counterclockwise order) are a saddle, an unstable node, a saddle, a saddle, a stable node and a saddle.

The vector field associated to system (2) on the invariant straight lines $x=0$ and $y=0$ is $\dot{x}|_{y=0} = ax(h-bx)$ and $\dot{y}|_{x=0} = y(h+by)$, respectively.

In the case (a) both finite saddles are forced to connect one stable separatrix with the finite unstable node, and one unstable separatrix with the finite stable node, the other separatrices of the saddles are forced to connect with the nodes at the infinity; and the separatrix of the infinite saddles, which are not at infinity, must connect to the finite nodes.

Note that in case (b), the three saddles are on $x = 0$ or $y = 0$, and some separatrices of these three saddles are in the boundary of the period annulus of the center, the other separatrices must connect with the nodes at the infinity.

Finally note that in the case (c), we put the attention in that the center is in a different quadrant with respect to the finite nodes, so the boundary of the period annulus of the center is formed by the separatrices of the saddle at the origin and the infinity, the other separatrices of the origin connect with the nodes contained in the axes.

Using the continuity of solutions with respect to initial conditions and parameters we can conclude that the phase portrait of system (2) for $h \neq 0$ in the Poincaré disc is topologically equivalent to one of the ones shown in (a), (b) or (c) of Figure 1.

3.1.2. *Subcase* $h = 0$. When $h = 0$ the finite singular points are reduced to a unique singular point, the origin $(0, 0)$, and in this case, the singular point is degenerate. System (2) for $h = 0$ is a quadratic homogeneous polynomial differential, then the results of section 4 in [7] presented in subsection 2.2 can be applied.

Following the notation of Proposition 7, we have that in this case $F(x, y) = (1 + ab - b)xy(x - y)$. Note that each invariant straight line divides the Poincaré disc in two regions, then as $F(x, y)$ has three invariant real linear factors, these are the straight lines $x = 0$, $y = 0$ and $x = y$, the Poincaré disc has six canonical regions. Therefore, from the infinite analysis done previously we have three possibilities for the local phase portraits of the infinite singular points of system (2). We recall that infinite singular points can be (in counterclockwise order) a stable node, a saddle, a stable node, an unstable node, a saddle, and an unstable node; or an unstable node, a stable node, an unstable node, a stable node, an unstable node and a stable node; or a saddle, an unstable node, a saddle, a saddle, a stable node and a saddle. Therefore the phase portraits of system (2) for $h = 0$ in the Poincaré disc are topologically equivalent to one of the ones shown in (f), (g) or (h) of Figure 1.

3.2. **Case** $b = 0$**.** Note that the finite singular points e_2 and e_3 of system (2) are not defined when $b = 0$. So we study this case separately using the system (3) in the plane $z = h$.

For $h = 0$ the invariant straight lines $x = 0$ and $y = 0$ are filled of singular points, and do not exist other finite singular points. Moreover the components of system (3) has the common factor xy, rescaling in the time by $xydt = ds$ systems (3) takes the form $x' = 1$, $y' = 1$, so all orbits are contained in the parallel straight lines with director vector $(1, 1)$. At infinity these parallel straight lines share a pair of diametrally opposite singular points. Therefore we can obtain the phase portrait of system (3) for $h = 0$ in the Poincaré disc having the axes filled of equilibria and changing the direction of the orbits on the parallel straight lines in the quadrants $x < 0$ and $y > 0$, and $x > 0$ and $y < 0$. This phase portrait is shown in Figure 1(l).

In the next we consider $h \neq 0$, and we separated the study in two subcases: $a \neq 0$ and $a = 0$.

3.2.1. *Subcase* $ha \neq 0$. Then we have two finite singular points, the origin and $(-h, -ah)$. The linear part (10) (with $b = 0$) at the origin has the eigenvalues ah and h, so it is an unstable node if $a > 0$ and $h > 0$, a stable node is $a > 0$ and $h < 0$, and a saddle if $a < 0$. Furthermore, for $a < 0$, if $h > 0$ the stable manifold of the saddle near to origin is located on the x-axis, and it is in the y-axis if $h < 0$. On the other hand, the linear part (10) of the system at $(-h, -ah)$ is $\begin{pmatrix} 0 & -h \\ -ah & 0 \end{pmatrix}$. So $(-h, -ah)$ is a saddle for $a > 0$, and a focus or a center if $a < 0$. But, note that at this singular point the first integral $H_3 = e^{x-y}x^h y^{-ah}$ (from Proposition 2) is well defined, thus we can affirm that this singular point is a center.

Note that if the origin is a saddle, then the other finite singular point is always a center. In the case that the origin is a node, then the other finite singular point is a saddle.

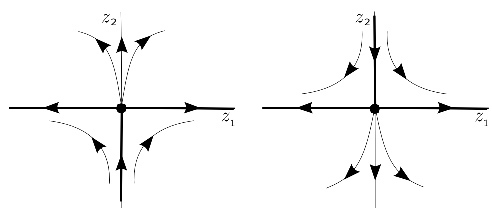
Now the infinite singular point $(1, 0)$ of U_1 has the eigenvalues both equals to -1, so always is a stable node.

On the other hand, the origin of the local charts U_1 and of U_2 are semi-hyperbolic singular points because the linear matrix (7) and (9) at the origin when $b = 0$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the notations and results of Theorem 2.19 of [8] applied to system (6) when $b = 0$ we obtain that the function $z_2 = f(z_1) = 0$ cancel the first component of the associated vector field, and then we obtain $g(z_1) = -ahz_1^2$. So the origin of U_1 is a saddle–node as in the Figure 5(a) if $-ah < 0$, and as in the Figure 5(b) if $-ah > 0$, in both cases the central manifold is the z_2–axis.

Applying Theorem 2.19 of [8] to the origin of U_2 we have that $z_2 = f(z_1) = 0$, then following the notations of that theorem $g(z_1) = -hz_1^2$. Hence the origin of U_2 is a saddle–node as it is shown in Figure 5 (this local phase portrait does not depend on the parameter a). In short, if $h \neq 0$ the origin of U_2 is a saddle–node where the central manifold is the z_2-axis.



(a) (b)

FIGURE 5. Local phase portrait at the semi–hyperbolic singular point $(0, 0)$ of system (6) with $b = 0$, when the associated function $g(z_1)$ of Theorem 2.19 on [8] is of the form $g(z_1) = kz_1^2$: (a) when $k > 0$, and (b) when $k < 0$.

To complete the phase portrait when $h \neq 0$ we note that for $a < 0$, the finite center is located in the quadrant $x < 0$ and $y > 0$, or $x > 0$ and $y < 0$. Then the boundary of its period annulus is formed by the infinity of this quadrant and the separatrices of the saddle at the origin connecting with the infinite saddle part of the saddle–nodes at the origins of V_1 and U_2, or U_1 and V_2 respectively. Since there are no others finite equilibria in the remaining quadrants we can complete the phase portrait considering the flow on the x and y. Both cases previously analyzed are topologically equivalent on the Poincaré disk.

In the case that the origin of system (3) is a node, we have that the finite saddle is in the quadrant $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$, where there is an infinite node which is not at the end of the axes, so the separatrices of the saddle must connect with the origin, with the mentioned infinite node and with the nodal parts of the saddle–nodes at the origins of U_1 and U_2. Since $x = 0$ and $y = 0$ are invariant we can complete the dynamics on the Poincaré disc. Again, these both cases are topologically equivalent on the Poincaré disc.

The possible phase portraits of system (3) when $ha \neq 0$ are shown in (d) and (e) of Figure 1.

3.2.2. *Subcase $h \neq 0$ and $a = 0$.* In this case system (2) writes

$$\dot{x} = xy, \quad \dot{y} = y(x + h). \tag{11}$$

Note that system (11) has the straight line $y = 0$ filled of equilibria, and $x = 0$ is an invariant straight line. We eliminate the common factor y doing a rescaling in the time and we get the system

$$\dot{x} = x, \quad \dot{y} = x + h. \tag{12}$$

This system is linear and it is easy study it. Note that does not have finite equilibria when $h \neq 0$, and the vector field in the x–direction increases in $x > 0$, and decreases in $x < 0$, while that in the y–direction the vector filed increases in $x > -h$, and decreases in $x < -h$. All orbits start in the origin of U_2 and end at infinity in the stables nodes located at U_1 and V_1. These orbits were studied in the previous subcase $ha \neq 0$ but now since we eliminate a common factor of degree one of the system, the maximum degree of the system decreases in one and the stability of V_1 is the same than in U_1, except for the orbit on $x = 0$ which remains invariant.

Taking into account the flow on the y-axis we can complete the phase portraits of system (12). These are shown in Figure 6.

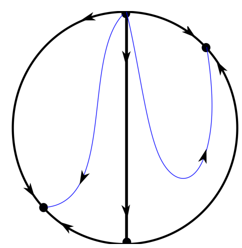


FIGURE 6. Phase portrait of system (12) when $h \neq 0$.

From the phase portrait of system (12) we can obtain the phase portrait of system (11), this is shown in Figure 1(k).

3.3. **Case** $b = 1/(1-a)$ **with** $a \neq 1$**.** For this special case as we have seen in section 3.1 the infinite singular points are not hyperbolic, they are semi–hyperbolic and the finite singular point e_4 does not exist. So we analyze this case separately.

Now system (2) takes the form

$$\dot{x} = x\left(ah + \frac{a}{a-1}x - \frac{1}{a-1}y\right), \quad \dot{y} = y\left(h + \frac{a}{a-1}x - \frac{1}{a-1}y\right). \tag{13}$$

As for the system (2) the axes are invariant under the flow.

If $h = 0$ system (13) has the common factor $(a/(a-1)x - 1/(a-1)y)$. If we eliminate this factor rescaling the time by $(a/(a-1)x - 1/(a-1)y)dt = ds$, the system has the vector field $\mathcal{X} = (x, y)$. Then the system has a unique finite singular point that is an unstable node at the origin, and its phase portrait is filled of orbits starting at the unstable node and ending at infinity. Thus we can give the phase portrait of system (13) for $h = 0$ adding the common factor and reversing the orbits where $(a/(a-1)x - 1/(a-1)y) < 0$. This phase portrait is shown in Figure 1(n).

If $h \neq 0$ system (13) has three equilibria: the origin $e_1 = (0,0)$, $e_2 = (0, h(a-1))$ and $e_3 = (h(1-a), 0)$. The linear part of system (13) was given in (10), but now with $b = 1/(1-a)$. Then the local phase portrait of these three singular points is described in Tables 1 and 2.

Note that in each case $a < 0$, or $0 < a < 1$, or $a > 1$, with $h \neq 0$, the finite singular points are a saddle, a stable node and an unstable node.

For studying the infinite singular points we consider the system in the local chart U_1, which is given by

$$\dot{z}_1 = -(-1+a)hz_1z_2, \quad \dot{z}_2 = -\frac{z_2}{a-1}(a - z_1 - ahz_2 + a^2hz_2).$$

Then $z_2 = 0$ is filled of singular points, i.e. the infinity is filled of singular points.

Taking into account the previous information, that the axes are invariant under the flow, and the continuity of the solutions with respect to the initial conditions and parameters, we can complete the phase portrait in the Poincarŕ disc. Thus for $h \neq 0$ this phase portraits is given in Figure 1(m).

3.4. **Case $a = 0$ and $b \neq 0$.** When $a = 0$ system (2) takes the form

$$\dot{x} = xy, \quad \dot{y} = y(h + x - bx + by). \tag{14}$$

We note that this system has the common factor y, so we eliminate it rescaling the time in order to analyze its dynamics in an easy way, and we get the system

$$\dot{x} = x, \quad \dot{y} = x + b(y - x) + h. \tag{15}$$

Note that $x = 0$ is invariant under the flow. System (15) has the unique finite singular point $(0, -h/b)$, with eigenvalues 1 and b, so it is an unstable node if $b > 0$, and it is a saddle if $b < 0$. Note that if $h = 0$ then the finite singular point is at the origin.

System (15) in the local chart U_1 writes

$$\dot{z}_1 = (-1 + b)(-1 + z_1) + hz_2, \quad \dot{z}_2 = -z_2.$$

At $z_2 = 0$ the unique infinite singular point is $(1, 0)$ with eigenvalues -1 and $b - 1$, so if $b < 1$ it is an unstable node, if $b > 1$ it is a saddle, and in the case $b = 1$ we have $\dot{z}_1 = hz_2$, then $z_2 = 0$ is filled of singular points, i.e. the infinity is filled of singular points.

The origin of U_2 is a singular point because the system in this chart is

$$\dot{z}_1 = z_1(1 - b + z_1(b - 1) - hz_2), \quad \dot{z}_2 = -z_2(z_1(1 - b) + hz_2 + b),$$

and the eigenvalues are $1 - b$ and $-b$, so we have that the origin is an unstable node if $b < 0$, a stable node if $b > 1$, and a saddle for $0 < b < 1$.

If the system has a finite saddle and since this is the unique finite singular point, its separatrices must connect with the infinite singular points. If the system has a finite unstable node the orbits starting in it must go to infinity.

Taking into account the flow on the invariant y-axis and the continuity of the solutions with respect to initial conditions and parameters, we can complete these phase portraits, which are shown in Figure 7.

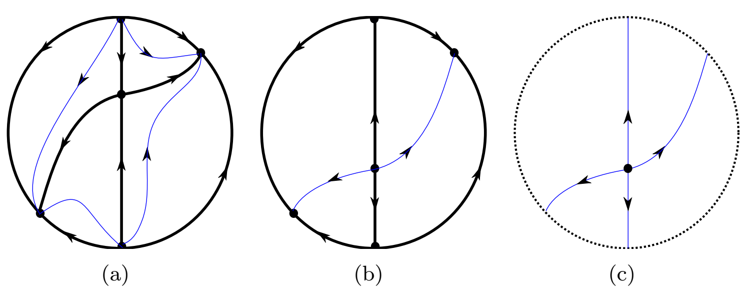


FIGURE 7. Phase portrait of system (15): (a) when $b < 0$, (b) when $0 < b < 1$ or $b > 1$, and (c) when $b = 1$. In these phase portrait we consider $h > 0$.

From the phase portraits of system (15) we can obtain the phase portraits of system (14). If $h \neq 0$ these are given in (i), (j) or (o) of Figure 1. As we already said if $h = 0$ the finite equilibria is at the origin (so it is on the straight line $y = 0$), then the phase portrait of system (13) when $h = 0$ is topologically equivalent to one of the phase portraits (p), (q), o (r) of Figure 1 .

This completes the proof of Theorem 5.

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