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DARBOUX INTEGRABILITY AND REVERSIBLE QUADRATIC VECTOR FIELDS

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ABSTRACT. In this paper we improve the Darboux theory of integrability for reversible polynomial vector fields in $\mathbf{R}^n$, and we classify the phase portraits of all $\varphi-$reversible quadratic polynomial vector fields of $\mathbf{R}^2$ such that the dimension of the set of fixed points of φ is equal to one.

1. Introduction and statement of the main results. The algebraic theory of integrability is a classical one. In 1878, Darboux [**11**] provided a link between algebraic geometry and the search of first integrals and showed how to construct the first integral of polynomial vector fields in $\mathbf{R}^2$ or $\mathbf{C}^2$ having sufficient invariant algebraic curves. The theory also received contributions from Poincaré [**24**], who mainly was interested in the rational first integrals.

Good extensions of the Darboux theory of integrability to polynomial systems in $\mathbf{R}^n$ or $\mathbf{C}^n$ are due to Jouanolou [**16**] and Weil [**29**], see also [**17**]. In [**4, 6–9**], the authors developed the Darboux theory of integrability essentially in $\mathbf{R}^2$ or $\mathbf{C}^2$ considering not only the invariant algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves. Recently, in [**13**] and [**18**] there are extensions of the Darboux theory of integrability to two-dimensional surfaces.

In this paper we present and prove properties of reversible polynomial vector fields. In Propositions 3 and 4 we prove that for $\varphi-$reversible polynomial vector fields, X, of degree greater than one and such that $\dim(\mathrm{Fix}(\varphi)) = k$, the involution φ is linear and conjugated to $\mathrm{diag}(+1, \dots, +1, -1, \dots, -1)$, where the number of -1 is equal to k. In Proposition 5 we prove that if $f = 0$ is an invariant curve of X, then $f \circ \varphi$ is also an invariant curve. The same occurs with the exponential

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