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Limit cycles of the generalized polynomial Liénard differential equations

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Abstract

We apply the averaging theory of first, second and third order to the class of generalized polynomial Liénard differential equations. Our main result shows that for any $n, m \geqslant 1$ there are differential equations of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with f and g polynomials of degree n and m respectively, having at least $[(n+m-1)/2]$ limit cycles, where $[\cdot]$ denotes the integer part function.

1. *Introduction*

The second part of the Hilbert's problem is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The generalized polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1·1}$$

was introduced in [**12**]. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively. For this subclass of polynomial vector fields we have a simplified version of Hilbert's problem, see [**13**] and [**22**].

In 1977 Lins, de Melo and Pugh [**13**] studied the classical polynomial Liénard differential equation (1·1), obtained when $g(x) = x$, and stated the following conjecture: *if* $f(x)$ *has degree* $n \geqslant 1$ *and* $g(x) = x$*, then* (1·1) *has at most* $[n/2]$ *limit cycles.* They also proved the conjecture for $n = 1, 2$. The conjecture for $n \in \{3, 4, 5\}$ is still open. For $n \geqslant 6$ this conjecture is not true as it has been proved recently by Dumortier, Panazzolo and Roussarie in [**5**].

We note that a classical polynomial Liénard differential equation has a unique singular point. However it is possible for generalized polynomial Liénard differential equations to have more than one singular point.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point,