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Darboux invariants for planar polynomial differential systems having an invariant conic

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Abstract. We characterize all the planar polynomial differential systems with a unique invariant algebraic curve given by a real conic and having a Darboux invariant.

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1. Introduction and statement of the main results

Real planar polynomial differential systems appear in many branches of applied mathematics, physics, and, in general, in applied sciences. For such differential systems, the existence of a first integral determines completely their phase portrait. The first integrals depending on the time, i.e., on the independent variable of the differential system, are called invariants. A special class of invariants is the Darboux invariants. As we shall see the invariants instead of determining the phase portrait of the system, we determine its $\alpha-$ and $\omega-$limit sets in the compactified polynomial differential system. That is, the Darboux invariants allow to describe the sets where all the orbits born or die.

In general, it is a very difficult problem to recognize when a given polynomial differential system in the plane has or not a first integral or a Darboux invariant. The goal of this paper is to classify all polynomial differential systems in the plane $\mathbb{R}^2$ having a Darboux invariant and a unique invariant algebraic curve given by a conic.

Let $\mathbb{K}[x, y]$ be the ring of the polynomials in the variables x and y with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We consider the *polynomial differential system* in $\mathbb{R}^2$ defined by

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, P and Q are relatively prime in $\mathbb{R}[x, y]$, and the dot denotes derivative with respect to the independent variable t usually called the *time*.

We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of system (1). We associate with system (1) the vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

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