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Quadratic polynomial differential systems in $\mathbb{R}^3$ having invariant planes with total multiplicity nine

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Abstract In this paper we consider all the quadratic polynomial differential systems in $\mathbb{R}^3$ having exactly nine invariant planes taking into account their multiplicities. This is the maximum number of invariant planes that these kind of systems can have, without taking into account the infinite plane. We prove that there exist thirty possible configurations for these invariant planes, and we study the realization and the existence of first integrals for each one of these configurations. We show that at least twenty three of these configurations are realizable and provide explicit examples for each one of them.

Keywords Polynomial differential systems · invariant planes · first integrals · extactic polynomial.

Mathematics Subject Classification (2000) MSC 58F14 · MSC 58F22 · MSC 34C05

1 Introduction and statement of the main results

Let $\mathbb{K}[x, y, z]$ be the ring of polynomials in the variables x, y and z with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Consider the polynomial differential

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system in $\mathbb{R}^3$ defined by

$$\dot{x} = P(x,y,z), \qquad \dot{y} = Q(x,y,z), \qquad \dot{z} = R(x,y,z), \tag{1}$$

where P, Q and R are relatively prime polynomials in $\mathbb{R}[x,y,z]$, and the dot denotes derivative with respect to the independent variable t, usually called the *time*.

We can associate to differential system (1) the vector field

$$\mathcal{X} = P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y} + R\,\frac{\partial}{\partial z}.$$

We say that $m = \max\{\deg(P), \deg(Q), \deg(R)\}$ is the *degree* of system (1) (or of the vector field $\mathcal{X}$). If $m = 2$ we say that system (1) is *quadratic*.

An *invariant algebraic surface* of differential system (1) or of the vector field $\mathcal{X}$ is an algebraic surface $f(x,y,z) = 0$, with $f \in \mathbb{C}[x,y,z] \setminus \mathbb{C}$, such that for some polynomial $K \in \mathbb{C}[x,y,z]$ we have $\mathcal{X}(f) = \langle (P,Q,R), \nabla f\rangle = Kf$, where ∇f denotes the gradient of the function f. The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$ and if m is the degree of the vector field $\mathcal{X}$, then the degree of K is at most $m-1$. If the polynomial f is irreducible in $\mathbb{C}[x,y,z]$, then we say that $f = 0$ is an *irreducible invariant algebraic surface*. Note that from this definition if an orbit of the vector field $\mathcal{X}$ has a point on $f = 0$, then the whole orbit is contained there, that is $f = 0$ is invariant by the flow of $\mathcal{X}$. If the degree of f is 1 then we say that the invariant algebraic surface $f = 0$ is an *invariant plane*.

A *first integral* of differential system (1) or of the vector field $\mathcal{X}$ in an open subset $U \subset \mathbb{R}^3$ is a non-constant analytic function $H : U \to \mathbb{R}$ which is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, that is $\mathcal{X}(H) = \langle (P,Q,R), \nabla H\rangle \equiv 0$ on U. Although system (1) is real, we allow complex invariant algebraic surfaces of this system. This is due to the fact that complex invariant algebraic surfaces play a role in the existence of real first integrals of system (1), because if $f = 0$ is an invariant algebraic surface of system (1) with cofactor K, then the conjugate complex surface $\overline{f} = 0$ is also an invariant algebraic surface of system (1) with cofactor $\overline{K}$, and the surface $f\,\overline{f} = 0$ is a real invariant algebraic surface of system (1) with cofactor $K\,\overline{K}$. For more details see Chapter 8 of [4].

The dynamics generated by the flow of the differential system (1) with degree $m \geq 2$ is, in general, very difficult to be studied. In this kind of differential systems are commonly encountered singular points, periodic, homoclinic and heteroclinic orbits, however, it may present more complicated dynamical behavior, as the occurrence of invariant tori, quasi-periodic orbits, strange or chaotic attractors and several other phenomena, see for instance [6,17,21]. The knowledge of invariant algebraic surfaces of system (1) provides important information for understanding its dynamics. The simplest use of invariant algebraic surfaces is for separating the phase space of system (1) into invariant pieces. If the number of invariant algebraic surfaces is sufficiently large, then they can also be used for computing explicitly first integrals, for more details see [1,7,16]. Furthermore, invariant algebraic surfaces allow to better control interesting regions, from the dynamical point of view, in the phase space of differential systems, as an example, in [5] the authors used invariant algebraic surfaces to provide a bounded region where the Lorenz attractor lives. More examples of the use of invariant algebraic surfaces in the study of the dynamical behavior of differential systems defined in $\mathbb{R}^3$ can be found in [9–13].

Let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x,y,z]$ such that $\dim(W)=N$ and $\{v_1,...,v_N\}$ be a basis of W. The *extactic polynomial of $\mathcal{X}$ associated to W* is the polynomial

$$\xi_W(\mathcal{X}) = \det\begin{pmatrix} v_1 & v_2 & \dots & v_N \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \dots & \mathcal{X}(v_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{N-1}(v_1) & \mathcal{X}^{N-1}(v_2) & \dots & \mathcal{X}^{N-1}(v_N) \end{pmatrix},$$

where $\mathcal{X}(v_i) = \langle (P,Q,R), \nabla v_i\rangle$ and $\mathcal{X}^{k+1}(v_i) = \mathcal{X}(\mathcal{X}^k(v_i))$, for $k=1,...,N-2$. If $W=\mathbb{R}_m[x,y,z]$, where $\mathbb{R}_m[x,y,z]$ is the $\mathbb{R}$–vector subspace of polynomials in $\mathbb{R}[x,y,z]$ of degree at most m, we say that the polynomial $\xi_W(\mathcal{X})$ is the *m–th extactic polynomial* of $\mathcal{X}$ and denote it by $\xi_W^m(\mathcal{X})$. The dimension of $\mathbb{R}_m[x,y,z]$ is $N=\binom{m+3}{3}$.

The notion of extactic polynomial already appears in the work of Lagutinskii, see [3] and references therein, and have been used as defined in [19] in different papers, see for instance [2,8,14,15,18]. In this work the definition of extactic polynomial and its properties play a fundamental role in the proof of the main theorem, as will be seen ahead.

Note that due to the properties of the determinant and of the derivation the definition of extactic polynomial is independent of the chosen basis of W, because for different bases the extactic polynomial differs by a non-zero constant.

We say that an irreducible invariant algebraic surface $f=0$ of degree m has *algebraic multiplicity* k if $\xi_W^m(\mathcal{X}) \not\equiv 0$ and k is the maximum positive integer such that f^k divides $\xi_W^m(\mathcal{X})$; and we say that it has *no defined algebraic multiplicity* if $\xi_W^m(\mathcal{X}) \equiv 0$.

Observe that the algebraic multiplicity of an invariant algebraic surface does not interfere in the dynamics of the differential system. However, if a differential system has an invariant algebraic surface of multiplicity k, a perturbation in this system can generate at most k distinct invariant algebraic surfaces. For instance, consider the quadratic differential system

$$\begin{aligned} \dot{x} &= P(x,y,z), \\ \dot{y} &= Q(x,y,z), \\ \dot{z} &= -z^2, \end{aligned} \tag{2}$$

where P and Q are polynomials of degree ≤ 2 in $\mathbb{R}[x,y,z]$. The plane $z=0$ is an invariant algebraic surface of system (2) of multiplicity 2. Given $\varepsilon>0$, if we perturb system (2) with the term ε^2, then $\dot{z}=-z^2+\varepsilon^2=(z+\varepsilon)(-z+\varepsilon)$ and from the invariant plane $z=0$ of multiplicity 2 we obtain two parallel invariant planes $z\pm\varepsilon=0$. Analogously, if we perturb system (2) with the term $\varepsilon^2 x^2$, then $\dot{z}=-z^2+\varepsilon^2 x^2=(z+\varepsilon x)(-z+\varepsilon x)$ and from the invariant plane $z=0$ of multiplicity 2 we obtain two concurrent invariant planes $z\pm\varepsilon x=0$.

In this paper we give all the possible configurations for quadratic polynomial differential systems of the form

$$\begin{aligned}
\dot{x} &= a_0+a_1\,x+a_2\,y+a_3\,z+a_4\,x^2+a_5\,y^2+a_6\,z^2+a_7\,x\,y+a_8\,x\,z+a_9\,y\,z,\\
\dot{y} &= b_0+b_1\,x+b_2\,y+b_3\,z+b_4\,x^2+b_5\,y^2+b_6\,z^2+b_7\,x\,y+b_8\,x\,z+b_9\,y\,z,\\
\dot{z} &= c_0+c_1\,x+c_2\,y+c_3\,z+c_4\,x^2+c_5\,y^2+c_6\,z^2+c_7\,x\,y+c_8\,x\,z+c_9\,y\,z,
\end{aligned} \tag{3}$$

where $a_i,b_i,c_i\in\mathbb{R}$, for $i=0,1,...,9$, having exactly nine invariant planes taking into account their multiplicities, which is the maximum number of invariant planes that differential system (3) can have, without taking into account the plane at infinity, as we shall prove in Proposition 3 of Sect. 2. We study the realization and the existence of first integrals for each one of the possible configurations of invariant planes for system (3).

We assume that differential system (3) has $k\leq 9$ distinct invariant planes $f_i=A_i\,x+B_i\,y+C_i\,z+D_i=0$, with $A_i,\,B_i,\,C_i,\,D_i\in\mathbb{R}$ and $A_i^2+B_i^2+C_i^2\neq 0$, for $i=1,...,k$, of multiplicity n_i such that $n_1+...+n_k=9$. Considering $f_1=A_1\,x+B_1\,y+C_1\,z+D_1=0$, without loss of generality, suppose $C_1\neq 0$. After the affine change of coordinates

$$(x,y,z)\to\left(x,y,-\frac{1}{C_1}(A_1\,x+B_1\,y-z+D_1)\right)$$

we can always consider $f_1=z=0$. Hence $c_0=c_1=c_2=c_4=c_5=c_7=0$ into system (3).

Here we say that differential system (3) has a *configuration of the form* $(n_1,...,n_k)$ if it has k invariant planes $f_1,...,f_k$ of multiplicities $n_1,...,n_k$, respectively. Our main result provides all the possible configurations for differential system (3) and a lower bound for the number of realizable ones.

Theorem 1 *The quadratic polynomial differential systems in $\mathbb{R}^3$ having exactly nine invariant planes taking into account their multiplicities admit thirty possible configurations of these planes, from which at least twenty three of them are realizable. In Table 1 are listed all possible configurations which these systems can admit and in Tables 2–4 we provide an example of quadratic polynomial differential system for twenty three of the thirty possible configurations.*

Theorem 1 is proved in Sect. 2. The examples of quadratic polynomial differential systems given in Tables 2–4 realizing the twenty three configurations are not unique. More precisely, for each configuration, it is possible to find other examples of quadratic polynomial differential systems different from those in Tables 2–4. Indeed, the differential system

$$\dot{x} = a_7\, xy + a_8\, xz,$$

$$\dot{y} = b_7\, xy + a_8\, yz,$$

$$\dot{z} = a_8\, z^2 + b_7\, xz + a_7\, yz,$$

with $a_7\, a_8\, b_7 \neq 0$, has the invariant planes $f_1 = z = 0$, $f_2 = x = 0$, $f_3 = y = 0$ and $f_4 = b_7\, x - a_7\, y = 0$ of multiplicity 4, 2, 2 and 1, respectively. Hence, it is another example for the configuration $(4, 2, 2, 1)$, different from that listed in Table 3.

Furthermore, Theorem 1 provides only a lower bound for the number of realizable configurations. In this way, there may be exist examples of quadratic polynomial differential systems for the other configurations which are not listed in Tables 2–4. However, finding these examples is a hard task due to the large number of parameters involved.

It is known that if a polynomial differential system has a sufficient number of invariant algebraic surfaces then it has a first integral. The following proposition provides all the configurations for differential system (3) given in Table 1 which have a first integral, taking into account only the number of invariant planes.

A *Darboux first integral* is a first integral of the form

$$\left(\prod_{i=1}^{r} f_i^{l_i} \right) \exp(g/h),$$

where f_i, g and h are polynomials, and the l_i's are complex numbers.

Proposition 1 *Consider all possible configurations for quadratic polynomial differential systems* (3) *in* $\mathbb{R}^3$ *having exactly nine invariant planes taking into account their multiplicities given in Table 1. Then the following statements hold.*

(i) *If some of these invariant planes has no defined algebraic multiplicity, then system* (3) *has a rational first integral.*

(ii) *Suppose that all invariant planes* $f_i = 0$ *has defined algebraic multiplicity* n_i *for* $i = 1, \ldots, k$. *If system* (3) *restricted to each plane* $f_i = 0$ *having multiplicity larger than* 1 *has no rational first integral, then the following statements hold.*

  (a) *If* $\sum_{i=1}^{k} n_i \geq 5$, *then system* (3) *has a Darboux first integral.*

  (b) *If* $\sum_{i=1}^{k} n_i \geq 7$, *then system* (3) *has a rational first integral.*

Proposition 1 is proved in Sect. 3.

The study of invariant algebraic surfaces is an important issue in Qualitative Theory of Differential Systems because they can be used as a tool to better understand differential systems that have complicated dynamics. Furthermore there is a lot of papers about differential systems in $\mathbb{R}^2$ having invariant algebraic curves, for example in [20] the authors characterize the class of quadratic polynomial differential systems in $\mathbb{R}^2$ having at least five invariant straight lines.

2 Proof of Theorem 1

Before proving Theorem 1 we show that, under the assumptions of that theorem, differential system (3) has the maximum number of invariant planes without taking into account the plane at infinity. The following proposition allows to detect when a plane $f = 0$ is an invariant plane for the vector field $\mathcal{X}$ and to compute easily its multiplicity. Since its proof is short, we present it here for the sake of completeness. We can also find a proof for it in [14]. A proof for this result in $\mathbb{C}^n$ can be found in [8].

Proposition 2 *Let $\mathcal{X}$ be a vector field on $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$, with $\dim(W) > 1$. Every invariant algebraic surface $f = 0$ of vector field $\mathcal{X}$, with $f \in W$, is a factor of the polynomial $\xi_W(\mathcal{X})$.*

Proof Let $f = 0$ be an invariant algebraic surface of $\mathcal{X}$ such that $f \in W$. As the definition of the extactic polynomial $\xi_W(\mathcal{X})$ is independent of the chosen basis of W, consider the basis $\{f, v_2, ..., v_N\}$ of W. In this case, we have

$$\xi_W(\mathcal{X}) = \det \begin{pmatrix} f & v_2 & ... & v_N \\ \mathcal{X}(f) & \mathcal{X}(v_2) & ... & \mathcal{X}(v_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{N-1}(f) & \mathcal{X}^{N-1}(v_2) & ... & \mathcal{X}^{N-1}(v_N) \end{pmatrix},$$

where $\mathcal{X}(f) = K\,f$, $\mathcal{X}^2(f) = (\mathcal{X}(K) + K^2)\,f$,..., $\mathcal{X}^{N-1}(f) = (\text{polynomial})\,f$. Then f is a factor of $\xi_W(\mathcal{X})$.

Note that the reciprocal of Proposition 2 is not true. That is if f is a factor of the extactic polynomial $\xi_W(\mathcal{X})$, with $f \in W$, then $f = 0$ is not necessarily an invariant algebraic surface of the vector field $\mathcal{X}$.

By Proposition 2, if $f = 0$ is an invariant plane of differential system (3), then the polynomial f is a factor of the extactic polynomial $\xi^1_W(\mathcal{X})$, since $f \in \mathbb{R}_1[x, y, z]$.

The next result provides, for differential system (3), the maximum number of invariant planes taking into account their multiplicities, the maximum number of distinct parallel invariant planes and the maximum number of distinct invariant planes passing through a single point. A proof for this result in $\mathbb{C}^n$ for differential systems of degree $m \geq 2$ can be found in [8]. For the sake of completeness we prove this result here for the differential systems of degree 2 in $\mathbb{R}^3$.

Proposition 3 *Assume that differential system* (3) *has finitely many invariant planes. Then the following statements hold.*

(a) The number of invariant planes of system (3) *taking into account their multiplicities is at most nine.*
(b) The number of different parallel invariant planes of system (3) *is at most two.*
(c) The number of different invariant planes of system (3) *through a single point is at most six.*

Moreover, all these upper bounds are reached.

Proof Let $f_i = 0$ be invariant planes of multiplicity n_i of differential system (3), for $i = 1, ..., k$. Note that $f_i \in \mathbb{R}_1[x, y, z]$ for each $i = 1, ..., k$, then, by the definition of algebraic multiplicity, $f_i^{n_i}$ is a factor of the extactic polynomial $\xi^1_W(\mathcal{X})$. Consider the basis $\{1, x, y, z\}$ of the $\mathbb{R}$–vector subspace $\mathbb{R}_1[x, y, z]$. The maximum degree of the polynomials $\mathcal{X}(v)$, $\mathcal{X}^2(v)$ and $\mathcal{X}^3(v)$, where $v \in \{1, x, y, z\}$, is 2, 3 and 4, respectively. By the definition of the determinant and the properties of operations with polynomials, it follows that the maximum degree of the polynomial $\xi^1_W(\mathcal{X})$ is 9. Then we can factor the extactic polynomial $\xi^1_W(\mathcal{X})$ in factors of the form $f_1^{n_1} \cdots f_k^{n_k}$ such that $n_1 + ... + n_k \leq 9$. This proves statement (a).

Now suppose that differential system (3) has k distinct parallel invariant planes. After an affine change of coordinates, we can assume that the equations of these invariant planes are of the form $f_i = z - \alpha_i$, where $\alpha_i \in \mathbb{R}$, for $i = 1, ..., k$. From the definition of invariant algebraic surface, we have that $\dot{z} = \beta\, f_1 \cdots f_k$ into system (3), where $\beta \in \mathbb{R}$. As system (3) has degree 2, then $k \leq 2$, which proves statement (b).

Finally suppose that $f_i = 0$ are distinct invariant planes of differential system (3) passing by a single point, for $i = 1, ..., k$. Doing a translation of the coordinates (if necessary), we can assume that this single point is the origin. So the equations of invariant planes $f_i = 0$ are of the form $f_i = A_i x + B_i y + C_i z$, where $A_i, B_i, C_i \in \mathbb{R}$, for $i = 1, ..., k$. Note that $f_i \in W$ for each $i = 1, .., k$, where W is the $\mathbb{R}$–vector subspace generated by $\{x, y, z\}$. Therefore, by Proposition 2, f_i is a factor of the extactic polynomial

$$\xi_W(\mathcal{X}) = \det \begin{pmatrix} x & y & z \\ \mathcal{X}(x) & \mathcal{X}(y) & \mathcal{X}(z) \\ \mathcal{X}^2(x) & \mathcal{X}^2(y) & \mathcal{X}^2(z) \end{pmatrix},$$

where $\mathcal{X}(v)$ and $\mathcal{X}^2(v)$, with $v \in \{x, y, z\}$, are polynomials of degrees 2 and 3, respectively. By the definition of the determinant and the properties of operations with polynomials, it follows that the maximum degree of the polynomial $\xi_W(\mathcal{X})$ is 6. As we are interested in the maximum number of distinct invariant planes $f_i = 0$, then their algebraic multiplicity is 1. Hence we can factor the extactic polynomial $\xi_W(\mathcal{X})$ in factors of the form $f_1 \cdots f_k$ with $k \leq 6$. Thus statement (c) is proved.

Moreover, all these upper bounds are reached. Indeed consider the differential system

$$\dot{x} = x + x^2, \qquad \dot{y} = y + y^2, \qquad \dot{z} = z + z^2, \tag{4}$$

Note that system (4) has nine invariant planes given by: $f_1 = x = 0$, $f_2 = y = 0$, $f_3 = z = 0$, $f_4 = x + 1 = 0$, $f_5 = y + 1 = 0$, $f_6 = z + 1 = 0$, $f_7 = x - y = 0$, $f_8 = x - z = 0$ e $f_9 = y - z = 0$, from which $f_1 = 0$ and $f_4 = 0$ are parallels (it is also the case of $f_2 = 0$ and $f_5 = 0$; $f_3 = 0$ and $f_6 = 0$) and $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, $f_7 = 0$, $f_8 = 0$ and $f_9 = 0$ pass by a single point, the origin.

In the following we prove our main result, that is Theorem 1.

Proof (Proof of Theorem 1) Suppose that differential system (3) has exactly nine invariant planes taking into account their multiplicities, that is $f_i = 0$ are invariant planes of multiplicity n_i of system (3), for $i = 1, ..., k \leq 9$, with $n_1 + ... + n_k = 9$.

We obtain all configurations $(n_1, ..., n_k)$ for system (3) as follows. We consider all combinations of digits $0, 1, ..., 9$ taking 9 of these digits (with possible repetitions) in such a way that their sum is 9. It is easy to check that there are 30 possible combinations. We omit all the zeros in the combinations, that is instead of writing $(8, 1, 0, 0, 0, 0, 0, 0, 0)$, for example, we simply write $(8, 1)$. Hence each obtained combination corresponds to a possible configuration of system (3), where the digits $1, ..., 9$ correspond to the multiplicities of the invariant planes. In Table 1 are listed all the possible configurations for system (3).

The examples of quadratic polynomial differential systems for each configuration provided in Tables 2–4 are obtained in the following way. As already observed in the Introduction, we can always consider $f_1 = z = 0$ as an invariant plane of system (3) and consequently $c_0 = c_1 = c_2 = c_4 = c_5 = c_7 = 0$. Hence we need to work with the 24 remaining parameters of system (3) in order to find the examples. Suppose that $f_1 = z = 0$ is an invariant plane of multiplicity n_1 of system (3), with $n_1 \leq 9$. From the definition of algebraic multiplicity, z^{n_1} must be a factor of the extactic polynomial $\xi^1_W(\mathcal{X})$, since all invariant planes of system (3) belong to $\mathbb{R}_1[x, y, z]$. Considering the basis $\{1, x, y, z\}$ of the $\mathbb{R}$–vector subspace $\mathbb{R}_1[x, y, z]$, then the extactic polynomial $\xi^1_W(\mathcal{X})$ is given by

$$\begin{aligned} \xi^1_W(\mathcal{X}) &= \mathcal{X}(x)\,\mathcal{X}^2(y)\,\mathcal{X}^3(z) + \mathcal{X}(y)\,\mathcal{X}^2(z)\,\mathcal{X}^3(x) + \mathcal{X}(z)\,\mathcal{X}^3(y)\,\mathcal{X}^2(x) \\ &\quad - \mathcal{X}(z)\,\mathcal{X}^2(y)\,\mathcal{X}^3(x) - \mathcal{X}^2(z)\,\mathcal{X}^3(y)\,\mathcal{X}(x) - \mathcal{X}^3(z)\,\mathcal{X}^2(x)\,\mathcal{X}(y) \\ &= \sum_{\substack{0 \leq i,j,l \leq 9 \\ i+j+l=9}} \alpha_{ijl}\, x^i\, y^j\, z^l, \end{aligned} \tag{5}$$

where α_{ijl} are real values which depend on the 24 parameters of the vector field $\mathcal{X}$ of system (3). In this case, we can easily check that z is always a factor of

$\xi^1_W(\mathcal{X})$. Assuming that the invariant plane $f_1 = z = 0$ has multiplicity $n_1 \geq 2$ we need to work with the 24 parameters of system (3) in order to obtain the factor z^{n_1} in the expression of the extactic polynomial $\xi^1_W(\mathcal{X})$. For example, consider the configurations (9) and $(8, 1)$ in Table 2.
- *Configuration* (9): We must have z^9 as a factor of the extactic polynomial $\xi^1_W(\mathcal{X})$. Let $r(x, y, z)$ be the remainder of the polynomial division of $\xi^1_W(\mathcal{X})$ by z^9. Observe that the coefficients of polynomial $r(x, y, z)$ depend on the 24 parameters of system (3). After working with these 24 parameters, one of the ways to obtain $r(x, y, z) \equiv 0$ is considering $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_7 = a_9 = 0$, $b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = 0$, $c_3 = c_8 = c_9 = 0$ and $b_9 = c_6 = a_8$. Thus, we have that z^9 is a factor of $\xi^1_W(\mathcal{X})$. Furthermore, it is necessary to consider $a_6\, a_8\, b_8 \neq 0$ because, otherwise, $\xi^1_W(\mathcal{X}) \equiv 0$ and the invariant plane has no defined algebraic multiplicity. The obtained system for these parameter values is the example for configuration (9) in Table 2.
- *Configuration* $(8, 1)$: We must have z^8 as a factor of $\xi^1_W(\mathcal{X})$. If $r(x, y, z)$ is the remainder of the polynomial division of $\xi^1_W(\mathcal{X})$ by z^8, after working with the 24 parameters of system (3) a possible way to obtain $r(x, y, z) \equiv 0$ is taking $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_7 = 0$, $b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = b_7 = b_8 = 0$, $c_3 = c_8 = c_9 = 0$, $b_6 = c_6 = a_8$ and $b_9 = 2a_8$. In this case, we must consider $a_8 \neq 0$ and $a_6 \neq a_9$, because, otherwise, $\xi^1_W(\mathcal{X}) \equiv 0$. Observe that, for this choice of parameters $y + z$ is also a factor of $\xi^1_W(\mathcal{X})$ and we can easily check that $f_2 = y + z = 0$ is an invariant plane of the obtained differential system. Then this system has two invariant planes $f_1 = z = 0$ and $f_2 = y + z = 0$ of multiplicities 8 and 1, respectively. This system is the example for configuration $(8, 1)$ in Table 2.

For the other configurations, when we consider z^{n_1} as a factor of the extactic polynomial $\xi^1_W(\mathcal{X})$ then we can write $\xi^1_W(\mathcal{X})$ of the form $\xi^1_W(\mathcal{X}) = z^{n_1}\, F(x, y, z)$, where F is a real polynomial of degree $9 - n_1$. Now we need to work with the remaining parameters of system (3) in order to write the polynomial F as the multiplication of factors of the form $f_2^{n_2} \cdots f_k^{n_k}$, where $f_2 = 0, ..., f_k = 0$ are the other invariant planes of system (3) of multiplicities $n_2, ..., n_k$, respectively, with $n_2 + ... + n_k = 9 - n_1$. As an example, consider the configuration $(7, 1, 1)$ in Table 2.
- *Configuration* $(7, 1, 1)$: We must have z^7 as a factor of $\xi^1_W(\mathcal{X})$. Let $r_1(x, y, z)$ be the remainder of the polynomial division of $\xi^1_W(\mathcal{X})$ by z^7. After working with the 24 parameters of system (3), a possible way to obtain $r_1(x, y, z) \equiv 0$ is taking $a_0 = a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_9 = 0$, $b_0 = b_1 = b_2 = b_4 = b_5 = b_6 = b_7 = 0$ and $c_3 = c_8 = c_9 = 0$. Hence the extactic polynomial $\xi^1_W(\mathcal{X})$ can be written of the form $\xi^1_W(\mathcal{X}) = z^7\, F(x, y, z)$, where F is a real polynomial of degree 2. Now, consider $r_2(x, y, z)$ as the remainder of the polynomial division of F by y. Note that $r_2(x, y, z)$ depends on the remaining parameters of system (3). After working with these parameters, observe that if we take $a_3 = b_3 = b_8 = 0$, then $r_2(x, y, z) \equiv 0$ and we can write $F(x, y, z) = \beta\, x\, y$, where $\beta \in \mathbb{R} \setminus \{0\}$. Consequently, x and y are factors of $\xi^1_W(\mathcal{X})$ and we can easily check that $f_2 = x = 0$ and $f_3 = y = 0$ are invariant planes of the obtained differential system. Moreover, we must consider $a_8\, b_9\, c_6 \neq 0$ and a_8, b_9 and c_6 pairwise

distinct, because otherwise $\xi_W^1(\mathcal{X}) \equiv 0$. Hence $f_1 = z = 0$, $f_2 = x = 0$ and $f_3 = y = 0$ are invariant planes of the obtained system of multiplicities 7, 1 and 1, respectively. This system is the example for configuration $(7, 1, 1)$ in Table 2.

In this way, after long and tedious calculations, we obtain all the examples given in Tables 2–4 for twenty three from the thirty possible configurations for system (3). Therefore, at least twenty three from the thirty possible configurations are realizable.

3 Proof of Proposition 1

From the next result (see Theorem 3 of [16]) if follows immediately Proposition 1.

Theorem 2 *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces.*

(*i*) *If some of these irreducible invariant algebraic hypersurfaces has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*

(*ii*) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ has defined algebraic multiplicity q_i for $i = 1, \ldots, p$. If $\mathcal{X}$ restricted to each hypersurface $f_i = 0$ having multiplicity larger than 1 has no rational first integral, then the following statements hold.*

  (*a*) *If $\sum_{i=1}^{p} q_i \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where $N = \binom{n+d-1}{n}$.*

  (*b*) *If $\sum_{i=1}^{p} q_i \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

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(9)	(4,2,2,1)
(8,1)	(4,2,1,1,1)
(7,2)	(4,1,1,1,1,1)
(7,1,1)	(3,3,3)
(6,3)	(3,3,2,1)
(6,2,1)	(3,3,1,1,1)
(6,1,1,1)	(3,2,2,2)
(5,4)	(2,2,2,1,1,1)
(5,3,1)	(3,2,2,1,1)
(5,2,2)	(3,2,1,1,1,1)
(5,2,1,1)	(3,1,1,1,1,1,1)
(5,1,1,1,1)	(2,2,2,2,1)
(4,4,1)	(2,2,1,1,1,1,1)
(4,3,2)	(2,1,1,1,1,1,1,1)
(4,3,1,1)	(1,1,1,1,1,1,1,1,1)

Table 1 All possible configurations for differential system (3) having exactly 9 invariant planes taking into account their multiplicities.

Configuration	**Example of quadratic polynomial differential system**	**Invariant plane (multiplicity)**
(9)	$\dot{x} = a_6 z^2 + a_8 xz$, $\dot{y} = b_8 xz + a_8 yz$, $\dot{z} = a_8 z^2$, with $a_6\, a_8\, b_8 \neq 0$.	$f_1 = z = 0 \quad (9)$
(8,1)	$\dot{x} = a_6 z^2 + a_8 xz + a_9 yz$, $\dot{y} = a_8 z^2 + 2\, a_8 yz$, $\dot{z} = a_8 z^2$, with $a_8 \neq 0$ and $a_6 \neq a_9$.	$f_1 = z = 0 \quad (8)$ $f_2 = y + z = 0 \quad (1)$
(7,2)	$\dot{x} = a_8 xz + a_9 yz$, $\dot{y} = a_8 yz$, $\dot{z} = c_6 z^2$, with $a_8\, a_9\, c_6 \neq 0$ and $a_8 \neq c_6$.	$f_1 = z = 0 \quad (7)$ $f_2 = y = 0 \quad (2)$
(7,1,1)	$\dot{x} = a_8 xz$, $\dot{y} = b_9 yz$, $\dot{z} = c_6 z^2$, with $a_8\, b_9\, c_6 \neq 0$, $a_8 \neq b_9$, $a_8 \neq c_6$ and $b_9 \neq c_6$.	$f_1 = z = 0 \quad (7)$ $f_2 = x = 0 \quad (1)$ $f_3 = y = 0 \quad (1)$
(6,3)	$\dot{x} = a_6 z^2 + a_8 xz$, $\dot{y} = a_8 yz$, $\dot{z} = a_8 z^2 + c_9 yz$, with $a_6\, a_8\, c_9 \neq 0$.	$f_1 = z = 0 \quad (6)$ $f_2 = y = 0 \quad (3)$
(6,2,1)	$\dot{x} = a_4 x^2 + 2a_8 xz$, $\dot{y} = a_4 xy - a_8 yz$, $\dot{z} = 2a_8 z^2$, with $a_4\, a_8 \neq 0$.	$f_1 = z = 0 \quad (6)$ $f_2 = x = 0 \quad (2)$ $f_3 = y = 0 \quad (1)$
(6,1,1,1)	$\dot{x} = a_8 xz$, $\dot{y} = -a_8 yz$, $\dot{z} = 2a_8 z^2 + c_8 xz$, with $a_8\, c_8 \neq 0$.	$f_1 = z = 0 \quad (6)$ $f_2 = x = 0 \quad (1)$ $f_3 = y = 0 \quad (1)$ $f_4 = c_8\, x + a_8\, z = 0 \quad (1)$
(5,3,1)	$\dot{x} = a_4 x^2 + a_8 xz$, $\dot{y} = a_4 xy$, $\dot{z} = a_8 z^2$, with $a_4\, a_8 \neq 0$.	$f_1 = z = 0 \quad (5)$ $f_2 = x = 0 \quad (3)$ $f_3 = y = 0 \quad (1)$
(5,2,2)	$\dot{x} = a_6 z^2 + a_7 xy + a_8 xz$, $\dot{y} = a_7 y^2$, $\dot{z} = a_8 z^2$, with $a_6\, a_7\, a_8 \neq 0$.	$f_1 = z = 0 \quad (5)$ $f_2 = y = 0 \quad (2)$ $f_3 = a_7\, y - a_8\, z = 0 \quad (2)$

Table 2 Examples of quadratic polynomial differential systems for the configurations (9), (8,1), (7,2), (7,1,1), (6,3), (6,2,1), (6,1,1,1), (5,3,1) and (5,2,2).

Configuration	**Example of quadratic polynomial differential system**	**Invariant plane (multiplicity)**
(5,2,1,1)	$\dot{x} = a_7xy + a_8xz$, $\dot{y} = a_7y^2$, $\dot{z} = c_6z^2$, with $a_7\, a_8\, c_6 \neq 0$ and $a_8 \neq \pm c_6$.	$f_1 = z = 0 \quad (5)$ $f_2 = y = 0 \quad (2)$ $f_3 = x = 0 \quad (1)$ $f_4 = a_7\, y - c_6\, z = 0 \quad (1)$
(4,4,1)	$\dot{x} = a_7xy$, $\dot{y} = 2\, c_6yz$, $\dot{z} = c_6z^2$, with $a_7\, c_6 \neq 0$.	$f_1 = y = 0 \quad (4)$ $f_2 = z = 0 \quad (4)$ $f_3 = x = 0 \quad (1)$
(4,3,2)	$\dot{x} = a_5y^2 + 2\, b_9xz$, $\dot{y} = b_9yz$, $\dot{z} = 2\, b_9z^2 + c_9yz$, with $a_5\, b_9\, c_9 \neq 0$.	$f_1 = y = 0 \quad (4)$ $f_2 = z = 0 \quad (3)$ $f_3 = c_9\, y + b_9\, z = 0 \quad (2)$
(4,3,1,1)	$\dot{x} = a_4x^2$, $\dot{y} = b_9yz$, $\dot{z} = -b_9z^2$ with $a_4\, b_9 \neq 0$.	$f_1 = z = 0 \quad (4)$ $f_2 = x = 0 \quad (3)$ $f_3 = y = 0 \quad (1)$ $f_4 = a_4\, x + b_9\, z = 0 \quad (1)$
(4,2,2,1)	$\dot{x} = a_7xz$, $\dot{y} = b_5y^2 + b_7xy + a_7yz$, $\dot{z} = a_7z^2 + b_7xz + 2b_5yz$, with $a_7\, b_5\, b_7 \neq 0$.	$f_1 = z = 0 \quad (4)$ $f_2 = y = 0 \quad (2)$ $f_3 = b_7\, x + b_5\, y = 0 \quad (2)$ $f_4 = x = 0 \quad (1)$
(4,2,1,1,1)	$\dot{x} = x + x^2$, $\dot{y} = y + y^2$ $\dot{z} = c_2\, y + z + c_2\, xy + yz$, with $c_2 \neq 0$.	$f_1 = x + 1 = 0 \quad (4)$ $f_2 = y = 0 \quad (2)$ $f_3 = x = 0 \quad (1)$ $f_4 = y + 1 = 0 \quad (1)$ $f_5 = x - y = 0 \quad (1)$
(4,1,1,1,1,1)	$\dot{x} = a_7xy + a_8xz$, $\dot{y} = b_2y + a_7y^2$, $\dot{z} = b_2z + c_6z^2$, with $a_7\, a_8\, b_2\, c_6 \neq 0$ and $a_8 \neq \pm c_6$.	$f_1 = z = 0 \quad (4)$ $f_2 = x = 0 \quad (1)$ $f_3 = y = 0 \quad (1)$ $f_4 = a_7\, y + b_2 = 0 \quad (1)$ $f_5 = c_6\, z + b_2 = 0 \quad (1)$ $f_6 = a_7\, y - c_6\, z = 0 \quad (1)$
(3,3,3)	$\dot{x} = a_6z^2$, $\dot{y} = b_5y^2 + b_9yz$, $\dot{z} = 2\, b_9z^2 + 2\, b_5yz$, with $a_6\, b_5\, b_9 \neq 0$.	$f_1 = y = 0 \quad (3)$ $f_2 = z = 0 \quad (3)$ $f_3 = b_5\, y + b_9\, z = 0 \quad (3)$

Table 3 Examples of quadratic polynomial differential systems for the configurations (5,2,1,1), (4,4,1), (4,3,2), (4,3,1,1), (4,2,2,1), (4,2,1,1,1), (4,1,1,1,1,1) and (3,3,3).

Configuration	**Example of quadratic polynomial differential system**	**Invariant plane (multiplicity)**
(3,3,2,1)	$\dot{x} = 2c_8xy + a_8xz$, $\dot{y} = 2c_8y^2$, $\dot{z} = c_8xz$, with $a_8\, c_8 \neq 0$.	$f_1 = y = 0 \quad (3)$ $f_2 = z = 0 \quad (3)$ $f_3 = x = 0 \quad (2)$ $f_4 = x - y = 0 \quad (1)$
(3,2,2,2)	$\dot{x} = a_4x^2 + a_7xy + a_8xz$, $\dot{y} = 2\,a_7y^2 + 2\,a_4xy$, $\dot{z} = 2\,a_7yz$, with $a_4\,a_7\,a_8 \neq 0$.	$f_1 = x = 0 \quad (3)$ $f_2 = y = 0 \quad (2)$ $f_3 = z = 0 \quad (2)$ $f_4 = a_4\,x + a_7\,y + a_8\,z = 0 \quad (2)$
(3,2,2,1,1)	$\dot{x} = a_7xy + a_8xz$, $\dot{y} = b_7xy$, $\dot{z} = 2a_8z^2 + b_7xz + a_7yz$, with $a_7\,a_8\,b_7 \neq 0$.	$f_1 = z = 0 \quad (3)$ $f_2 = x = 0 \quad (2)$ $f_3 = a_7\,y + 2\,a_8\,z = 0 \quad (2)$ $f_4 = y = 0 \quad (1)$ $f_5 = b_7\,x + a_8\,z = 0 \quad (1)$
(3,2,1,1,1,1)	$\dot{x} = a_7xy$, $\dot{y} = b_2y - a_7y^2$, $\dot{z} = b_2z + c_6z^2$, with $a_7\,b_2\,c_6 \neq 0$.	$f_1 = y = 0 \quad (3)$ $f_2 = z = 0 \quad (2)$ $f_3 = x = 0 \quad (1)$ $f_4 = a_7\,y - b_2 = 0 \quad (1)$ $f_5 = c_6\,z + b_2 = 0 \quad (1)$ $f_6 = a_7\,y + c_6\,z = 0 \quad (1)$
(2,2,2,1,1,1)	$\dot{x} = a_4x^2$, $\dot{y} = b_5y^2$, $\dot{z} = c_6z^2$, with $a_4\,b_5\,c_6 \neq 0$.	$f_1 = x = 0 \quad (2)$ $f_2 = y = 0 \quad (2)$ $f_3 = z = 0 \quad (2)$ $f_4 = a_4\,x - b_5\,y = 0 \quad (1)$ $f_5 = a_4\,x - c_6\,z = 0 \quad (1)$ $f_6 = b_5\,y - c_6\,z = 0 \quad (1)$
(1,1,1,1,1,1,1,1,1)	$\dot{x} = a_1x + a_4x^2$, $\dot{y} = a_1y + b_5y^2$, $\dot{z} = a_1z + c_6z^2$, with $a_1\,a_4\,b_5\,c_6 \neq 0$.	$f_1 = x = 0 \quad (1)$ $f_2 = y = 0 \quad (1)$ $f_3 = z = 0 \quad (1)$ $f_4 = a_4\,x + a_1 = 0 \quad (1)$ $f_5 = b_5\,y + a_1 = 0 \quad (1)$ $f_6 = c_6\,z + a_1 = 0 \quad (1)$ $f_7 = a_4\,x - b_5\,y = 0 \quad (1)$ $f_8 = a_4\,x - c_6\,z = 0 \quad (1)$ $f_9 = b_5\,y - c_6\,z = 0 \quad (1)$

Table 4 Examples of quadratic polynomial differential systems for the configurations (3,3,2,1), (3,2,2,2), (3,2,2,1,1), (3,2,1,1,1,1), (2,2,2,1,1,1) and (1,1,1,1,1,1,1,1,1).