International Journal of Bifurcation and Chaos, Vol. 20, No. 10 (2010) 3137–3155

DOI: 10.1142/S0218127410027593

GLOBAL DYNAMICS OF THE LORENZ SYSTEM WITH INVARIANT ALGEBRAIC SURFACES

JAUME LLIBRE
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
jllibre@mat.uab.cat

MARCELO MESSIAS
Departamento de Matemática, Estatística e Computação, Faculdade de Ciências e Tecnologia, UNESP – Univ Estadual Paulista, Cx. Postal 266, 19060-900, Presidente Prudente, São Paulo, Brazil
marcelo@fct.unesp.br

PAULO RICARDO DA SILVA
Departamento de Matemática, Instituto de Biociências, Letras e Ciências Exatas, UNESP – Univ Estadual Paulista, Rua C. Colombo, 2265, 15054–000, S. J. Rio Preto, São Paulo, Brazil
prs@ibilce.unesp.br

Received December 22, 2009; Revised February 4, 2010

In this paper by using the Poincaré compactification of $\mathbb{R}^3$ we describe the global dynamics of the Lorenz system

$$\dot{x} = s(-x + y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy,$$

having some invariant algebraic surfaces. Of course $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(s, r, b) \in \mathbb{R}^3$ are the parameters. For six sets of the parameter values, the Lorenz system has invariant algebraic surfaces. For these six sets, we provide the global phase portrait of the system in the Poincaré ball (i.e. in the compactification of $\mathbb{R}^3$ with the sphere $\mathbb{S}^2$ of the infinity).

Keywords: Integrability; Lorenz system; Poincaré compactification; dynamics at infinity; invariant algebraic surface.

1. Introduction and Main Results

In this work, we do a global analysis of the Lorenz system, given by

$$\begin{aligned} \dot{x} &= P(x, y, z) = s(-x + y), \\ \dot{y} &= Q(x, y, z) = rx - y - xz, \\ \dot{z} &= R(x, y, z) = -bz + xy, \end{aligned} \tag{1}$$

where the state variables are $(x, y, z) \in \mathbb{R}^3$ and s, r and b are real parameters. As usual, the dots denote derivative with respect to the independent variable, or the time t. System (1) was proposed by the meteorologist Lorenz in 1963 in the study of thermal fluid convection in the atmosphere, concerning the question of long-term weather forecast, see [Lorenz, 1963]. There are hundreds of papers concerning the rich dynamical behavior of system (1), see for instance [Sparrow, 1982] and the more recent work [Viana, 2000] for good reviews on the subject. Most of them consider strictly positive values for the parameters s, r and b due to their physical meaning. Here, we consider $(s, r, b) \in \mathbb{R}^3$,