DARBOUX THEORY OF INTEGRABILITY FOR REAL POLYNOMIAL VECTOR FIELDS ON $\mathbb{S}^n$

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Abstract. This is a survey on the Darboux theory of integrability for polynomial vector fields, first in $\mathbb{R}^n$ and second in the n-dimensional sphere $\mathbb{S}^n$. We also provide new results about the maximum number of invariant parallels and meridians that a polynomial vector field $\mathcal{X}$ on $\mathbb{S}^n$ can have in function of its degree. These results in some sense extend the known result on the maximum number of hyperplanes that a polynomial vector field $\mathcal{Y}$ in $\mathbb{R}^n$ can have in function of the degree of $\mathcal{Y}$.

1. Introduction and statement of the main results

The real nonlinear ordinary differential systems are used to model a wide range of processes practically all fields of science, from biology and chemistry to economy, physics and engineering. The existence of a first integrals of differential systems defined on $\mathbb{R}^n$ is important for two main things. First, they make easier the characterization the phase portrait of the system. Secondly, their existence allow reducing the dimension of the system by one, which in many cases makes easier the analysis. In our terminology, a system is *integrable* if it has a first integral. Therefore, the methods to detect the presence of first–integrals and find their explicit form are extremely important in the qualitative theory of differential equations.

However, as in many cases occurs, the more important the problem is, the more difficult are the ways to access it. The techniques to finding the presence and constructing first–integrals goes back to Darboux, as far as we know [11]. Hamiltonians are the vector fields whose first–integral are easiest to find. If the integrable vector fields are not Hamiltonian, various techniques have been developed for analyzing the existence of first integrals, such as the Noether symmetries [5], the already mentioned Darbouxian theory of integrability [11] or the Lie symmetries [27]. In fact, Emmy Noether's Theorems represent a relevant example of the interdisciplinary character acquired by the problem of finding first integrals. By (roughly speaking), stating that any physical conservation law has its associated symmetry, it establishes a connection between mechanics, Lie algebra and differential equations. Other contributions to this problem are represented by the Painlevé analysis [3], the use of Lax pairs [17] or the direct method [14] and [15], to cite only a few of them.

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