IMPROVING THE AVERAGING THEORY FOR COMPUTING PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATIONS

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ABSTRACT. For $m = 1, 2, 3$, we consider differential systems of the form

$$x' = F_0(t, x) + \sum_{i=1}^{m} \varepsilon^i F_i(t, x) + \varepsilon^{m+1} R(t, x, \varepsilon),$$

where $F_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}^n$, and $R : \mathbb{R} \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^{m+1}$ functions, and T–periodic in the first variable, being $\mathcal{D}$ an open subset of $\mathbb{R}^n$, and ε a small parameter. For such system we assume that the unperturbed system $x' = F_0(t, x)$ has a k–dimensional manifold of periodic solutions with $k \leq n$. We weaken the sufficient assumptions for studying the periodic solutions of the perturbed system when $|\varepsilon| > 0$ is sufficiently small.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The averaging theory for computing periodic solutions of a differential system is one of the best analytical tools for the study of the periodic solutions, see for instance the papers [23, 13, 19, 12, 14, 17, 18, 20, 10, 11, 8].

In the analysis of periodic solutions using the averaging theory there are essentially three main theorems. One for studying the periodic solutions of the periodic differential systems of the form $x' = \varepsilon F(t, x, \varepsilon)$, with $x \in \mathbb{R}^n$ (see for instance [26, 24, 2, 9, 15, 16]), and the other two for studying the periodic solutions of the periodic differential systems of the form $x' = F_0(t, x) + \varepsilon F(t, x, \varepsilon)$ with $x \in \mathbb{R}^n$, distinguishing when the manifold $\mathcal{Z}$ of all periodic solutions of the unperturbed system $x' = F_0(t, x)$ has dimension n or smaller then n, respectively (see for instance [21, 22, 3, 4, 9, 15, 16]). The objective of this paper is to weaken the sufficient assumptions under the last system for studying their periodic solutions when the dimension of the manifold $\mathcal{Z}$ is smaller then n.

Let $\mathcal{D}$ be an open set of $\mathbb{R}^n$ and $\varepsilon_0 > 0$ be a small parameter. For $m = 1, 2, 3$, we assume that the $\mathcal{C}^{m+1}$ functions $F_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}^n$ for $i = 1, \ldots, m$ and $R : \mathbb{R} \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are T–periodic in the variable t. In what follows the prime denotes derivative with respect to the time t. Our study is concerned

about the existence of isolated T–periodic solutions of the differential systems of the form

$$(1) \qquad x' = F_0(t, x) + \sum_{i=1}^{m} \varepsilon^i F_i(t, x) + \varepsilon^{m+1} R(t, x, \varepsilon),$$

when there exists a k–dimensional submanifold $\mathcal{Z}$ of $\mathcal{D}$ ($k \leq n$) such that all solutions of the unperturbed system

$$(2) \qquad x' = F_0(t, x),$$

starting in $\mathcal{Z}$ are T–periodic. Formally, let $\beta : \overline{V} \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^{m+1}$ function, with V an open and bounded subset of $\mathbb{R}^k$ and let $\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\}$. We shall assume that

(H) $\mathcal{Z} \subset \mathcal{D}$ and for each $z_\alpha \in \mathcal{Z}$ the unique solution $x_\alpha(t) = x(t, z_\alpha, 0)$ of (2) such that $x_\alpha(0) = z_\alpha$ is T–periodic.

Here as usual $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^n$ denotes the solution of system (1) such that $x(0, z, \varepsilon) = z$.

Let $Y(t, z)$ be a fundamental matrix solution of the linear differential system

$$(3) \qquad y' = \frac{\partial F_0}{\partial x}(t, x(t, z, 0))\, y.$$

For each $z_\alpha \in \mathcal{Z}$ we denote $Y_\alpha(t) = Y(t, z_\alpha)$. System (3) is the linearization of system (2) through the solution $x(t, z, 0)$. For a given fundamental matrix solution $Y(t, z)$ we denote by Γ_α the upper right corner $k \times (n - k)$ matrix of $Y_\alpha(0)^{-1} - Y_\alpha(T)^{-1}$, and by Δ_α the lower right corner $(n - k) \times (n - k)$ matrix of $Y_\alpha(0)^{-1} - Y_\alpha(T)^{-1}$.

Malkin [21] and Rosseau [22] have studied this problem up to order 1 in ε. Buică, Françoise and Llibre gave in [3] a simpler proof of their results when Γ_α is a null matrix for every $\alpha \in \mathcal{Z}$. Buică, Giné and Llibre in [4] have studied this problem up to order 2 in ε also assuming that Γ_α is a null matrix for every $\alpha \in \mathcal{Z}$, and in [5] the same authors provided a nondegeneracy condition assuring the existence of a fundamental matrix solution $Y(t, z)$ for which Γ_α is a null matrix for every $\alpha \in \mathcal{Z}$. They also gave the formulae of the averaged function up to order 2 in ε for this case. In [7] Rhouma and Chicone studied the same problem when the unperturbed system is autonomous.

In this work we give the formulae of the averaged functions up to order 3 in ε without assuming that Γ_α is a null matrix. This approach represents a new way to deal with this kind of problem avoiding a previous treatment of the system. Moreover, as far as we know, the explicit formulae for the third order averaged function for systems like (1) are given for the first time in this present paper.

Choosing a system of coordinates where the matrix $\dfrac{\partial F_0}{\partial x}(t, x_\alpha(t))$ is in its Jacobian normal form for every t we obtain a system for which Γ_α is a null matrix.

Unfortunately it is not always possible, as we can see in (9). Nevertheless it does not mean that the matrix Γ_α is different from zero for every system of coordinates and for every election of fundamental matrix solution $Y(t, z)$.

Given p and q positive integers let $G = (G_1, G_2, \ldots, G_q) : \mathbb{R}^p \to \mathbb{R}^q$ be a smooth function, and consider the vectors $u = (u_1, u_2, \cdots, u_p) \in \mathbb{R}^p$ and $v^i = (v_1^i, v_2^i, \cdots, v_p^i) \in \mathbb{R}^p$ for $i = 1, 2, 3$. Thus for each $t \in \mathbb{R}$ and $\overline{u} \in \mathbb{R}^p$ we define

$$\frac{\partial^2 G}{\partial u^2}(t, \overline{u})v^1 v^2 = \left(\sum_{i,j=1}^p \frac{\partial^2 G_1}{\partial u_i \partial u_j}(t, \overline{u})v_i^1 v_j^2 \,, \; \cdots \,, \; \sum_{i,j=1}^p \frac{\partial^2 G_q}{\partial u_i \partial u_j}(t, \overline{u})v_i^1 v_j^2 \right),$$

$$(4) \quad \frac{\partial^3 G}{\partial u^3}(t, \overline{u})v^1 v^2 v^3 = \left(\sum_{i,j,k=1}^p \frac{\partial^2 G_1}{\partial u_i \partial u_j \partial u_k}(t, \overline{u})v_i^1 v_j^2 v_k^3 \,, \; \cdots \,, \right.$$

$$\left. \sum_{i,j=1}^p \frac{\partial^2 G_q}{\partial u_i \partial u_j \partial u_k}(t, \overline{u})v_i^1 v_j^2 v_k^3 \right).$$

We note that for each $t \in \mathbb{R}$ and $\overline{u} \in \mathbb{R}^p$, $\dfrac{\partial^2 G}{\partial u^2}(t, \overline{u}) : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ is a 2–linear map and $\dfrac{\partial^3 G}{\partial u^3}(t, \overline{u}) : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ is a 3–linear map.

In what follows the functions $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $\pi^\perp : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ will denote the projections onto the first k coordinates and onto the last $n - k$ coordinates, respectively. For a point $z \in \mathcal{D}$ we also consider $z = (a, b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

We define the averaged functions $f_1, f_2, f_3 : \overline{V} \to \mathbb{R}^k$ as

$$f_1(\alpha) = \Gamma_\alpha \gamma_1(\alpha) + \pi g_1(z_\alpha),$$

$$f_2(\alpha) = \frac{1}{2} \Gamma_\alpha \gamma_2(\alpha) + \frac{1}{2} \frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi g_2(z_\alpha),$$

$$(5) \quad f_3(\alpha) = \frac{1}{6} \Gamma_\alpha(z_\alpha)\gamma_3(\alpha) + \frac{1}{6} \frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + \frac{1}{2} \frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\gamma_2(\alpha)$$

$$+ \frac{1}{2} \frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{1}{2} \frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_2(\alpha) + \frac{\partial \pi g_2}{\partial b}(z_\alpha)\gamma_1(\alpha)$$

$$+ \pi g_3(z_\alpha),$$

where

$$\gamma_1(\alpha) = -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha),$$

$$\gamma_2(\alpha) = -\Delta_\alpha^{-1} \left(\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha)\gamma_1(\alpha) + 2\pi^\perp g_2(z_\alpha) \right),$$

$$\text{(6)} \quad \gamma_3(\alpha) = -\Delta_\alpha^{-1} \left(\frac{\partial^3 \pi^\perp g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + 3\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\gamma_2(\alpha) \right.$$

$$+3\frac{\partial^2 \pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha)\gamma_2(\alpha) + 6\frac{\partial \pi^\perp g_2}{\partial b}(z_\alpha)\gamma_1(\alpha)$$

$$\left. +6\pi^\perp g_3(z_\alpha) \right),$$

and

$$\text{(7)} \qquad\qquad g_i(z) = Y(T, z)^{-1}\frac{y_i(T, z)}{i!},$$

for $i = 0, 1, 2, 3$, being

$$y_0(t, z) = x(t, z, 0) - z,$$

$$y_1(t, z) = Y(t, z)\int_0^t Y(\tau, z)^{-1}F_1(\tau, x(\tau, z, 0))d\tau,$$

$$y_2(t, z) = Y(t, z)\int_0^t Y(\tau, z)^{-1}\left[2F_2(\tau, x(\tau, z, 0)) + 2\frac{\partial F_1}{\partial x}(\tau, x(\tau, z, 0)) \right.$$

$$\left. \cdot y_1(\tau, z) + \frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0))y_1(\tau, z)^2 \right] d\tau,$$

$$\text{(8)}$$

$$y_3(t, z) = Y(t, z)\int_0^t Y(\tau, z)^{-1}\left[6F_3(\tau, x(\tau, z, 0)) \right.$$

$$+6\frac{\partial F_2}{\partial x}(\tau, x(\tau, z, 0))y_1(\tau, z) + 3\frac{\partial^2 F_1}{\partial x^2}(\tau, x(\tau, z, 0))y_1(\tau, z)^2$$

$$+3\frac{\partial F_1}{\partial x}(\tau, x(\tau, z, 0))\, y_2(\tau, z) + 3\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0))y_1(\tau, z)$$

$$\left. \cdot y_2(\tau, z) + \frac{\partial^3 F_0}{\partial x^3}(\tau, x(\tau, z, 0))y_1(\tau, z)^3 \right] d\tau.$$

Following the notation (4) the expression $\dfrac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2$ in (5), is obtained taking $p = n - k$, $q = k$, $u = b$, $\bar{u} = \beta(\alpha)$, $G : b \mapsto \pi g_0(\alpha, b)$, and $v^1 = v^2 = \gamma_1(\alpha)$. The similar expressions (6) and (8) are obtained analogously.

Our main result is given in the following theorem.

Theorem 1. *In addition to hypothesis (H) we assume that for each $z_\alpha \in \mathcal{Z}$, there exists a fundamental matrix solution of (3) such that $\det(\Delta_\alpha) \neq 0$ for every $\alpha \in \overline{V}$. So the following statements hold.*

(a) *Take m=1. If there exists $\alpha^* \in V$ with $f_1(\alpha^*) = 0$ and $\det(Df_1(\alpha^*)) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

(b) *Take m=2 and assume that $f_1(\alpha) \equiv 0$. If there exists $\alpha^* \in V$ with $f_2(\alpha^*) = 0$ and $\det(Df_2(\alpha^*)) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

(c) *Take m=3 and assume that $f_1(\alpha) \equiv f_2(\alpha) \equiv 0$. If there exists $\alpha^* \in V$ with $f_3(\alpha^*) = 0$ and $\det(Df_3(\alpha^*)) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

Theorem extends the main results of [4, 5], and it is proved in section 2.

As an application of Theorem 1 we consider the following quadratic polynomial non–autonomous system of differential equations:

$$u' = w + \varepsilon\, \mu\, v^2 \sin t + \varepsilon^2 \nu,$$

$$(9) \qquad v' = w + \varepsilon\, \eta\, v^2 \sin t + \varepsilon^2 \nu,$$

$$w' = (1 + \cos t)w + \varepsilon^2 \bar{\nu} + \varepsilon^3 \xi \sin t.$$

Let

$$A_0(t) = \int_0^t e^{s + \sin s} ds, \quad \text{and} \qquad A_1(t) = \int_0^t e^{-s - \sin s} A_0(s) \sin s\, ds.$$

We assume that

$$(10) \qquad \bar{\nu} = \frac{2\pi(e^{2\pi} - 1)\,\nu}{(e^{2\pi} - 1)A_1(2\pi) - A_0(2\pi)A_0(-2\pi)} \approx 0.775\,\nu.$$

We note that for $\varepsilon = 0$ the linear part of system (9) is given by the matrix

$$(11) \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 + \cos t \end{pmatrix}.$$

For $t \neq \pi$ the Jacobian normal form of (11) is given by

$$
(12) \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 + \cos t \end{pmatrix}.
$$

Nevertheless the Jacobian normal form of (11) when $t = \pi$ is given by

$$
(13) \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Hence there is no continuous change of variables which transforms the matrix (11) into (12).

In the next proposition we give sufficient conditions for the existence of an isolated periodic solution of system (9) when $|\varepsilon| > 0$ is sufficiently small.

Proposition 2. *If $\mu\,\eta\,\nu \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exist an isolate periodic solution $\varphi(t, \varepsilon)$ of system (9) such that $\varphi(0, \varepsilon) \to (u^*, v^*, 0) \in \mathcal{D}$ where*

$$
(u^*, v^*) = k \left(\frac{\xi}{\eta\,\nu}, \frac{\xi}{\mu\,\nu} \right) \quad and \quad k \approx 2.952.
$$

2. PROOFS OF THEOREM 1 AND PROPOSITION 2

We shall need the next lemma to prove our main result. Lemma 3 extends Theorem 4 of [4] which is a particular case of the *Lyapunov–Schmidt reduction* method for finite dimensional function (see for instance [6]).

Lemma 3. *Assume that $k \leq n$ are positive integers. Let $\mathcal{D}$ and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. For $m = 1, 2, 3$, let $g_i(z)$ for $i = 0, 1, \ldots, m$ and $\beta : \overline{V} \to \mathbb{R}^{n-k}$ be $\mathcal{C}^{m+1}$ functions and take $g : \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ such that*

$$
(14) \qquad g(z, \varepsilon) = g_0(z) + \sum_{i=1}^{m} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{m+1}),
$$

and $\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\} \subset \mathcal{D}$. We denote by Γ_α the upper right corner $k \times (n - k)$ matrix of $D\,g_0(z_\alpha)$, and by Δ_α the lower right corner $(n - k) \times (n - k)$ matrix of $D\,g_0(z_\alpha)$. Assume that for each $z_\alpha \in \mathcal{Z}$, $\det(\Delta_\alpha) \neq 0$ and $g_0(z_\alpha) = 0$. We consider the functions $f_1, f_2, f_3 : \overline{V} \to \mathbb{R}^k$ defined in (5). Then the following statements hold.

> *(a) Take $m = 1$. If there exists $\alpha^* \in V$ with $f_1(\alpha^*) = 0$ and $\det(D\,f_1(\alpha^*)) \neq 0$, then there exists α_ε such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and $z_{\alpha_\varepsilon} \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

(b) *Take $m = 2$ and assume that $f_1(\alpha) \equiv 0$. If there exists $\alpha^* \in V$ with $f_2(\alpha^*) = 0$ and $\det(D f_2(\alpha^*)) \neq 0$, then there exists α_ε such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and $z_{\alpha_\varepsilon} \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

(c) *Take $m = 3$ and assume that $f_1(\alpha) \equiv f_2(\alpha) \equiv 0$. If there exists $\alpha^* \in V$ with $f_3(\alpha^*) = 0$ and $\det(D f_3(\alpha^*)) \neq 0$, then there exists α_ε such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and $z_{\alpha_\varepsilon} \to z_{\alpha^*}$ when $\varepsilon \to 0$.*

Note that in the Lemma 3 the functions g_i for $i = 0, 1, 2, 3$, which appear in the expression of (5) and (6) are the ones of the function (14), instead of the functions which appear in (7).

Proof of Lemma 3. First of all we consider $g = (\pi g, \pi^\perp g)$, $g_i = (\pi g_i, \pi^\perp g_i)$ for $i = 0, 1, 2$, and $z = (a, b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ for $z \in \mathcal{D}$. So

$$\frac{\partial g}{\partial z}(z_\alpha, 0) = D g_0(z_\alpha) = \begin{pmatrix} \dfrac{\partial \pi g_0}{\partial a}(z_\alpha) & \dfrac{\partial \pi g_0}{\partial b}(z_\alpha) \\ \dfrac{\partial \pi^\perp g_0}{\partial a}(z_\alpha) & \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha) \end{pmatrix}.$$

We note that $\Gamma_\alpha = \dfrac{\partial \pi g_0}{\partial b}(z_\alpha)$ and $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$.

From the hypotheses we have $\pi^\perp g(\alpha, \beta(\alpha), 0) = \pi^\perp g_0(z_\alpha) = 0$ and

$$\det\left(\frac{\partial \pi^\perp g}{\partial b}(\alpha, \beta(\alpha), 0)\right) = \det\left(\frac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)\right) = \det(\Delta_\alpha) \neq 0.$$

Thus applying the Implicit Function Theorem (see for instance [25]) it follows that there exists an open neighbourhood $U \times (-\varepsilon_1, \varepsilon_1)$ of $\overline{V} \times \{0\}$ with $\varepsilon_1 \leq \varepsilon_0$, and a $\mathcal{C}^{m+1}$ function $\overline{\beta} : U \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^{n-k}$ such that $\pi^\perp g(a, \overline{\beta}(a, \varepsilon), \varepsilon) = 0$ for each $(a, \varepsilon) \in U \times (-\varepsilon_1, \varepsilon_1)$ and $\overline{\beta}(\alpha, 0) = \beta(\alpha)$ for every $\alpha \in \overline{V}$. Here the value

of m depends on the statement (a), (b) or (c) we are proving. So we compute

$$
\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, 0) = -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) = \gamma_1(\alpha),
$$

$$
\frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, 0) = -\Delta_\alpha^{-1} \left(\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha) \gamma_1(\alpha)^2 \right.
$$

$$
\left. +2 \frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha) \gamma_1(\alpha) + 2\pi^\perp g_2(z_\alpha) \right) = \gamma_2(\alpha),
$$

(15)

$$
\frac{\partial^3 \overline{\beta}}{\partial \varepsilon^2}(\alpha, 0) = -\Delta_\alpha^{-1} \left(\frac{\partial^3 \pi^\perp g_0}{\partial b^3}(z_\alpha) \gamma_1(\alpha)^3 + 3 \frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha) \gamma_1(\alpha) \gamma_2(\alpha) \right.
$$

$$
+3 \frac{\partial^2 \pi^\perp g_1}{\partial b^2}(z_\alpha) \gamma_1(\alpha)^2 + 2 \frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha) \gamma_2(\alpha)
$$

$$
\left. +6 \frac{\partial \pi^\perp g_2}{\partial b}(z_\alpha) \gamma_1(\alpha) + 6\pi^\perp g_3(z_\alpha) \right) = \gamma_3(\alpha).
$$

Now for each $\alpha \in \overline{V}$ we consider the $\mathcal{C}^{m+1}$ function $\delta : \varepsilon \mapsto \pi g(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \in \mathbb{R}^k$. We know that

$$
(16) \quad \delta(\alpha, \varepsilon) = \delta(\alpha, 0) + \varepsilon \frac{\partial \delta}{\partial \varepsilon}(\alpha, 0) + \varepsilon^2 \frac{1}{2} \frac{\partial^2 \delta}{\partial \varepsilon^2}(\alpha, 0) + \varepsilon^3 \frac{1}{6} \frac{\partial^3 \delta}{\partial \varepsilon^3}(\alpha, 0) + \mathcal{O}(\varepsilon^4).
$$

We observe that the above expression is written assuming that $m = 3$. If $m = 2$ it must be truncated in $\mathcal{O}(\varepsilon^2)$, and if $m = 1$ it must be truncated in $\mathcal{O}(\varepsilon^2)$.

We compute

$$
\frac{\partial \delta}{\partial \varepsilon}(\alpha, \varepsilon) = \frac{\partial \pi g}{\partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) + \frac{\partial \pi g}{\partial \varepsilon}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon),
$$

$$
\frac{\partial^2 \delta}{\partial \varepsilon^2}(\alpha, \varepsilon) = \frac{\partial^2 \pi g}{\partial b^2}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \right)^2
$$

$$
+2 \frac{\partial^2 \pi g}{\partial \varepsilon \partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) + \frac{\partial \pi g}{\partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, \varepsilon)
$$

$$
+ \frac{\partial^2 \pi g}{\partial \varepsilon^2}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon),
$$

$$\frac{\partial^3 \delta}{\partial^3 \varepsilon}(\alpha, \varepsilon) = \quad \frac{\partial^3 \pi g}{\partial b^3}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \right)^3 + 3 \frac{\partial^2 \pi g}{\partial b^2}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon)$$

$$\cdot \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, \varepsilon) + \frac{\partial \pi g}{\partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial^3 \overline{\beta}}{\partial \varepsilon^3}(\alpha, \varepsilon)$$

$$+ 3 \frac{\partial^3 \pi g}{\partial \varepsilon \partial b^2}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \right)^2 + 3 \frac{\partial^2 \pi g}{\partial \varepsilon \partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon)$$

$$\cdot \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, \varepsilon) + 3 \frac{\partial^3 \pi g}{\partial \varepsilon^2 \partial b}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon)$$

$$+ \frac{\partial^3 \pi g}{\partial \varepsilon^3}(\alpha, \overline{\beta}(\alpha, \varepsilon), \varepsilon).$$

Thus

$$\delta(\alpha, 0) = \quad 0,$$

$$\frac{\partial \delta}{\partial \varepsilon}(\alpha, 0) = \quad \frac{\partial \pi g_0}{\partial b}(z_\alpha) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, 0) + \pi g_1(z_\alpha)$$

$$= \quad \Gamma_\alpha \gamma_1(\alpha) + \pi g_1(z_\alpha) = f_1(\alpha),$$

$$\frac{\partial^2 \delta}{\partial \varepsilon^2}(\alpha, 0) = \quad \frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, 0) \right)^2 + 2 \frac{\partial \pi g_1}{\partial b}(z_\alpha) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, 0)$$

$$+ \frac{\partial \pi g_0}{\partial b}(z_\alpha) \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, 0) + 2 \pi g_2(z_\alpha)$$

$$= \quad \Gamma_\alpha \gamma_2(\alpha) + \frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha) \gamma_1(\alpha)^2 + 2 \frac{\partial \pi g_1}{\partial b}(z_\alpha) \gamma_1(\alpha)$$

$$+ 2 \pi g_2(z_\alpha) = 2 f_2(\alpha),$$

$$\frac{\partial^3 \delta}{\partial^3 \varepsilon}(\alpha, \varepsilon) = \quad \frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \right)^3 + 3 \frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, \varepsilon)$$

$$+ \frac{\partial \pi g_0}{\partial b}(z_\alpha) \frac{\partial^3 \overline{\beta}}{\partial \varepsilon^3}(\alpha, \varepsilon) + 3 \frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha) \left(\frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) \right)^2 + 3 \frac{\partial \pi g_1}{\partial b}(z_\alpha)$$

$$\cdot \frac{\partial^2 \overline{\beta}}{\partial \varepsilon^2}(\alpha, \varepsilon) + 6 \frac{\partial \pi g_2}{\partial b}(z_\alpha) \frac{\partial \overline{\beta}}{\partial \varepsilon}(\alpha, \varepsilon) + 6 \pi g_3(z_\alpha)$$

$$
\begin{aligned}
= \;& \Gamma_\alpha(z_\alpha)\gamma_3(\alpha) + \frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + 3\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\gamma_2(\alpha) \\
& +3\frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 3\frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_2(\alpha) + 6\frac{\partial \pi g_2}{\partial b}(z_\alpha)\gamma_1(\alpha) \\
& +6\pi g_3(z_\alpha) = 6f_3(\alpha).
\end{aligned}
$$

If $m = 1$ and $f_1(\alpha) \not\equiv 0$, then the Implicit Function Theorem, applied to the function $\delta(\alpha, \varepsilon)/\varepsilon$ for $|\varepsilon| > 0$ sufficiently small, guarantees the existence of $\alpha(\varepsilon)$ such that $\alpha(0) = \alpha^*$ and $\delta(\alpha(\varepsilon), \varepsilon) = 0$. Denoting $z_{\alpha_\varepsilon} = (\alpha(\varepsilon), \overline{\beta}(\alpha(\varepsilon)), \varepsilon)$ we have $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$. So statement (a) is proved.

If $m = 2$ and $f_1(\alpha) \equiv 0$, then the Implicit Function Theorem, applied to the function $\delta(\alpha, \varepsilon)/\varepsilon^2$ for $|\varepsilon| > 0$ sufficiently small, guarantees the existence of $\alpha(\varepsilon)$ such that $\alpha(0) = \alpha^*$ and $\delta(\alpha(\varepsilon), \varepsilon) = 0$. Denoting $z_{\alpha_\varepsilon} = (\alpha(\varepsilon), \overline{\beta}(\alpha(\varepsilon)), \varepsilon)$ we have $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$. So statement (b) is proved.

Finally If $m = 3$ and $f_1(\alpha) \equiv f_2(\alpha) \equiv 0$, then the Implicit Function Theorem, applied to the function $\delta(\alpha, \varepsilon)/\varepsilon^3$ for $|\varepsilon| > 0$ sufficiently small, guarantees the existence of $\alpha(\varepsilon)$ such that $\alpha(0) = \alpha^*$ and $\delta(\alpha(\varepsilon), \varepsilon) = 0$. Denoting $z_{\alpha_\varepsilon} = (\alpha(\varepsilon), \overline{\beta}(\alpha(\varepsilon)), \varepsilon)$ we have $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$. So statement (c) is proved. $\qquad\square$

Proof of Theorem 1. For $z \in \mathcal{D}$ we consider $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^n$ the solution of (1) such that $x(0, z, \varepsilon) = z$. From hypothesis (H), $t_{(z_\alpha,0)} > T$ for every $\alpha \in V$. So as a consequence of Theorem 8.3 of [1] there exists a neighborhood of U of $\mathcal{Z}$ and $0 < \varepsilon_2 < \varepsilon_1$ such that, for all $(z, \varepsilon) \in U \times (-\varepsilon_2, \varepsilon_2)$, $t_{(z,\varepsilon)} > T$. Hence we can consider the function $f : U \times (-\varepsilon_2, \varepsilon_2) \to \mathbb{R}^n$, given by

$$
(17) \qquad\qquad\qquad f(z, \varepsilon) = x(T, z, \varepsilon) - z.
$$

It is easy to see that system (1) for $\varepsilon = \bar{\varepsilon} \in (-\varepsilon_2, \varepsilon_2)$ has a periodic solution passing through $\bar{z} \in U$ if and only if $f(\bar{z}, \bar{\varepsilon}) = 0$.

To study the zeros of (17) is equivalent to study the zeros of the $\mathcal{C}^{m+1}$ function

$$
(18) \qquad\qquad\qquad g(z, \varepsilon) = Y(T, z)^{-1} f(z, \varepsilon).
$$

On the other hand, we have that

$$
(19) \qquad x(t, z, \varepsilon) = x(t, z, 0) + \varepsilon y_1(t, z) + \varepsilon^2 \frac{y_2(t, z)}{2} + \varepsilon^3 \frac{y_3(t, z)}{6} + \mathcal{O}(\varepsilon^4),
$$

where y_1, y_2 and y_3 are given in (8). For more details on these expression (19) see Lemma 1 of [15, 16]. Thus

(20)
$$
\begin{aligned}
g(z,\varepsilon) \ =\ & Y(T,z)^{-1}(x(t,z,0)-z) + \varepsilon Y(T,z)^{-1}y_1(T,z) + \varepsilon^2 Y(T,z)^{-1}\frac{y_2(T,z)}{2} \\
& +\varepsilon^3 Y(T,z)^{-1}\frac{y_3(t,z)}{6} + \mathcal{O}(\varepsilon^4) \\
=\ & g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \varepsilon^3 g_3(z) + \mathcal{O}(\varepsilon^4).
\end{aligned}
$$

The same observation made about the value of m for the expression (16) is applied for the expressions (19) and (20).

In order to apply Lemma 3 to function (18) we compute

$$
g_0(z_\alpha) = Y_\alpha(T)^{-1}(x(T,z_\alpha,0) - z_\alpha) = 0,
$$

and

$$
\begin{aligned}
\frac{\partial g_0}{\partial z}(z_\alpha) =\ & Y_\alpha(T)^{-1}\left(\frac{\partial x}{\partial z}(T,z_\alpha,0) - Id\right) \\
=\ & Y_\alpha(T)^{-1}\left(Y_\alpha(T)Y_\alpha(0)^{-1} - Id\right) \\
=\ & Y_\alpha(0)^{-1} - Y_\alpha(T)^{-1},
\end{aligned}
$$

because $(\partial x/\partial z)(t,z,0)$ is solution of (3) such that $(\partial x/\partial z)(0,z,0) = Id$, which implies that $(\partial x/\partial z)(t,z,0) = Y(t,z)Y(0,z)^{-1}$.

From here the proof follows applying Lemma 3. □

Proof of Proposition 2. First of all we define the following functions,

$$
A_0(t) = \int_0^t e^{s+\sin s}ds, \ \ A_1(t) = \int_0^t e^{-s-\sin s}A_0(s)\,ds, \ \ A_2(t) = \int_0^t A_0(s)\sin s\,ds,
$$

$$
A_3(t) = \int_0^t A_0(s)^2 \sin s\,ds, \ \ A_4(t) = \int_0^t A_0(s)\cos s \sin s\,ds,
$$

$$
A_5(t) = \int_0^t A_0(s)A_2(s)\sin s\,ds, \ \ A_6(t) = \int_0^t A_0(s)A_3(s)\sin s\,ds,
$$

$$
A_7(t) = \int_0^t A_2(s)\sin s\,ds, \ \ A_8(t) = \int_0^t A_3(s)\sin s\,ds,
$$

$$
A_9(t) = \int_0^t A_0(s)A_0(-s)\sin s\,ds, \ \ A_{10}(t) = \int_0^t e^{-s-\sin s}A_0(s)\sin^2 s\,ds,
$$

$$
A_{11}(t) = \int_0^t A_1(s)\sin^2 s\,ds, \ \ A_{12}(t) = \int_0^t e^{-s-\sin s}\sin^2 s\,ds.
$$

Now we shall describe the different elements which appear in the statement of Theorem 1 in the particular case of the differential system (9). Firstly, $T = 2\pi$, $\mathcal{D} = \mathbb{R}^3$, $x = (u, v, w)$, $F_0(t, x) = (w\,,\,w\,,\,(1 + \cos\,t)w)$, $F_1(t, x) = (\mu\,v^2\sin t\,,\,\eta\,u^2$ $\sin t\,,\,0)$, $F_2(t, x) = (\nu\,,\,\nu\,,\,\overline{\nu})$, and $F_3(t, x) = (0\,,\,0\,,\,\xi\sin t)$. The solution of the unperturbed system starting in $z = (u_0, v_0, w_0)$ is given by

$$x(t, z, 0) = \big(u_0 + w_0 A_0(t)\,,\,v_0 + w_0 A_0(t)\,,\,w_0\,e^{t + \sin\,t}\big).$$

Thus for $w_0 = 0$, $x(t, u_0, v_0, 0, 0) = (u_0\,,\,v_0\,,\,0)$ is a T–periodic solution. So $\alpha = (u_0, v_0)$, $\beta(\alpha) \equiv 0$, $k = 2$ and $n = 3$. Now we take the parameters $r_1 > 0$ and $r_2 > 0$ arbitrarily small and arbitrarily large, respectively. Let V be the open and bounded subset of the plane $w_0 = 0$ given by

$$V = \{(u_0, v_0, 0) \in \mathbb{R}^3 : r_1 < \sqrt{u_0^2 + v_0^2} < r_2\}.$$

Therefore, in our case the set

$$\mathcal{Z} = \big\{z_\alpha = (\alpha, \beta(\alpha))\,,\,\alpha \in \overline{V}\big\} = \{(u_0, v_0, 0) \in \mathbb{R}^3 : r_1 \leq \sqrt{u_0^2 + v_0^2} \leq r_2\}.$$

Taking $Y_\alpha(t) = \dfrac{\partial x}{\partial z}(t, z_\alpha, 0)$ we obtain that

$$Y_\alpha(0)^{-1} - Y_\alpha(2\pi)^{-1} = \begin{pmatrix} 0 & 0 & e^{-2\pi}A_0(2\pi) \\ 0 & 0 & e^{-2\pi}A_0(2\pi) \\ 0 & 0 & 1 - e^{2\pi} \end{pmatrix}.$$

So

$$\Gamma_\alpha = \begin{pmatrix} e^{-2\pi}A_0(2\pi) \\ e^{-2\pi}A_0(2\pi) \end{pmatrix} \quad \text{and} \quad \Delta_\alpha = 1 - e^{2\pi}.$$

We note that Γ_α and Δ_α do not depend on α, and it is easy to check that they are not null matrices.

Using the formulae (7) and (8) we compute

$$g_0(u, v, w) = \big(e^{-2\pi}wA_0(2\pi)\,,\,e^{-2\pi}wA_0(2\pi)\,,\,w\,(1 - e^{-2\pi})\big)$$

and

$$y_1(t, z) = \Big(\mu\big(2vwA_2(t) + w^2A_3(t) + v^2(1 - \cos t)\big)\,,\,\eta\big(2uwA_2(t)$$
$$+ w^2 A_3(t) + u^2(1 - \cos t)\big)\,,\,0\Big).$$

So

$$g_1(z) = \Big(\mu\,w\big(2vA_2(2\pi) + wA_3(2\pi)\big)\,,\,\eta\,w\big(2uA_2(2\pi) + wA_3(2\pi)\big)\,,\,0\Big).$$

Using the formulae (5) and (6) we obtain that $\gamma_1(u,v) = 0$ and $f_1(u,v) = (0,0)$. In this case the expressions of γ_2 and f_1 become simpler.

(21) $\qquad f_2(\alpha) = \dfrac{1}{2}\Gamma_\alpha \gamma_2(\alpha) + \pi g_2(z_\alpha) \quad \text{and} \quad \gamma_2(\alpha) = -2\Delta_\alpha^{-1}\pi^\perp g_2(z_\alpha).$

Again from (8) we compute $y_2(t,z) = (y_2^1(t,z)\,,\, y_2^2(t,z)\,,\, y_2^3(t,z))$ where

$$
\begin{aligned}
y_2^1(t,z) = \ & \nu\, t - \overline{\nu}(A_0(t)A_0(-t) + A_1(t)) + 4\mu\nu\, u^2 v \sin^4 \frac{t}{2} + 2\mu\eta\, w\Big(u^2(A_2(t) \\
& - A_4(t)) + 2uw A_5(t) + w^2 A_6(t) + 2uv A_7(t) + tvw A_8(t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
y_2^1(t,z) = \ & \nu\, t - \overline{\nu}(A_0(t)A_0(-t) + A_1(t)) + 4\mu\nu\, u\, v^2 \sin^4 \frac{t}{2} + 2\mu\eta\, w\Big(v^2(A_2(t) \\
& - A_4(t)) + 2vw A_5(t) + w^2 A_6(t) + 2uv A_7(t) + tuw A_8(t)\Big),
\end{aligned}
$$

$$
y_2^3(t,z) = \ -\overline{\nu}e^{t+\sin t} A_0(-t).
$$

So from (7)

$$
\begin{aligned}
g_2(z) = \ & \Big(\nu\pi - \frac{\overline{\nu}}{2}A_1(2\pi) + \mu\eta\, w\Big(u^2(A_2(2\pi) - A_4(2\pi)) + 2uw A_5(2\pi) \\
& + w^2 A_6(2\pi) + 2uv A_7(2\pi) + 2\pi vw A_8(2\pi)\Big)\,,\, \nu\pi - \frac{\overline{\nu}}{2}A_1(2\pi) \\
& + \mu\eta\, w\Big(v^2(A_2(2\pi) - A_4(2\pi)) + 2vw A_5(2\pi) + w^2 A_6(2\pi) \\
& + 2uv A_7(2\pi) + 2\pi uw A_8(2\pi)\Big)\,,\, -\frac{\overline{\nu}}{2}A_0(-2\pi)\Big).
\end{aligned}
$$

Using now the formulae (21) we obtain

$$
\gamma_2(u,v) = \frac{\overline{\nu}A_0(-2\pi)}{1 - e^{-2\pi}},
$$

and

$$
\begin{aligned}
f_2(u,v) = \ & \Big(\pi\nu + \frac{\overline{\nu}A_0(2\pi)A_0(-2\pi)}{2(e^{2\pi} - 1)} - \frac{\overline{\nu}A1(2\pi)}{2}, \\
& \pi\nu + \frac{\overline{\nu}A_0(2\pi)A_0(-2\pi)}{2(e^{2\pi} - 1)} - \frac{\overline{\nu}A1(2\pi)}{2}\Big) = (0\,,\, 0).
\end{aligned}
$$

The last equality was obtained by substituting the value of $\overline{\nu}$ given in (10).

Now using the formulae (8) and (6) we compute, respectively,

$$g_3(u, v, 0) = \Big(-\xi A_{10}(2\pi) - \mu\, v(2\pi\nu + \overline{\nu}(A_9(2\pi) + A_{11}(2\pi))), \ -\xi A_{10}(2\pi)$$

$$-\mu\, u(2\pi\nu + \overline{\nu}(A_9(2\pi) + A_{11}(2\pi))), \ \xi A_{12}(2\pi)\Big),$$

$$\gamma_3(u, v) = \frac{6\xi A_{12}(2\pi)}{e^{-2\pi} - 1}.$$

So from (5) we obtain

$$f_3(u, v) = M \begin{pmatrix} u \\ v \end{pmatrix} - \xi\, \chi,$$

where

$$M = \begin{pmatrix} 0 & 2\pi\mu\nu\, K \\ 2\pi\eta\nu\, K & 0 \end{pmatrix}, \ \chi = \begin{pmatrix} A_{10}(2\pi) + \dfrac{A_0(2\pi)A_{12}(2\pi)}{e^{2\pi} - 1} \\ A_{10}(2\pi) + \dfrac{A_0(2\pi)A_{12}(2\pi)}{e^{2\pi} - 1} \end{pmatrix} \approx \begin{pmatrix} 4.178 \\ 4.178 \end{pmatrix},$$

and

$$K = \frac{(e^{2\pi} - 1)(A_1(2\pi) + A_9(2\pi) + A_{11}(2\pi)) - (A_0(2\pi) + e^{2\pi}A_2(2\pi))A_0(-2\pi)}{A_0(2\pi)A_0(2\pi) + (e^{2\pi} - 1)A_1(2\pi)}$$

$$\approx 0.225.$$

From the hypotheses we have that $\det M \neq 0$. Hence there exists a solution (u^*, v^*) of the linear system $f_3(u, v) = 0$ such that $\det(Df_3(u^*, v^*)) \neq 0$. The proof follows by computing $(u^*, v^*) = M^{-1}\chi$ and applying Theorem (1). $\qquad\square$

Acknowledgements

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