PERIODIC SOLUTIONS OF THE NATHANSON'S AND THE COMB-DRIVE FINGER MODELS

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Abstract. We provide sufficient conditions for the existence and stability of periodic solutions of the second–order non–autonomous differential equation of the Nathanson's model

$$\ddot{x} + x + a\dot{x} - \frac{b(v_0 + \delta v(\omega t))^2}{(1-x)^2} = 0,$$

and of the comb-drive finger model

$$\ddot{x} + x + a\dot{x} - \frac{4b(v_0 + \delta v(\omega t))^2 x}{(1-x^2)^2} = 0,$$

where $x \in \mathbb{R}$, c, β, v_0 and δ are positive parameters, $v(\omega t)$ is a $2\pi/\omega$–periodic function. The results are obtained using the averaging theory.

1. Introduction and statement of the main results

In Micro-Electro-Mechanical Systems (MEMS) the *electrostatic actuators* are the most used devices due to its wide variety of applications, mainly in sensing and actuation. Because the cheap production and the high performance of this technology, MEMS devices can be found all around us nowadays, for example in crash air-bag deployment systems in cars, gyroscopes for smart-phones, kidney dialysis to monitor the inlet and outlet pressures of blood (see [13]), but also in several resonant sensors [1], accelerometers [7], micro-pumps [17] and micro-valves [15]. An excellent review of the many applications of these devices can be found in [21, 22] and the references there in. In the last several years a large number of electrostatic actuators have been studied both from the numerical [5, 6, 19] and experimental point of view, [14, 19]. However, the number of documents devoted to a rigorously mathematical analysis of these devices is relatively low. In order to understand the dynamics of this devices, the first approach was introduced into the literature by Nathanson in [14] (1967) by means of a "lumped" mass-spring model were the elastic behaviour of the system is modelled by a linear spring and the electrostatic forces are computed considering of a simple parallel plate capacitor.

Up to our knowledge, the first mathematical analysis of periodic solutions for the *Nathanson's model* appears in [2] (2007) by means of shooting techniques and later in [9] (2013) using degree theory and lower and upper solution method where the authors prove the existence, multiplicity and stability of periodic solutions. Recently in [8] (2017) using the Leray-Schauder continuation theorem, the authors

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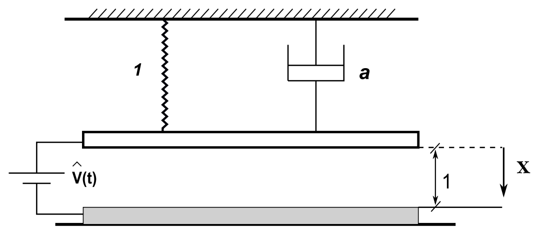

FIGURE 1. Idealized Nathanson's model with spring force $\mathbf{x}$, damping force $a\dot{\mathbf{x}}$, voltage source $\hat{V}(t)$ and initial gap between the moveable top electrode and the bottom fixed electrode is 1.

prove the existence of symmetric periodic solutions for other electrostatic actuator known in the literature as a *Comb-drive finger*. Motivated by the successful use of well-developed mathematical techniques for the study of some canonical MEMS, the aim of this document is provide sufficient conditions for the existence and stability of periodic solutions of the equation of motions of two special types of electrostatic actuators, namely, the Nathanson's model (or Parallel-Plate capacitor model) which is based on the resonant gate transistor [14] and the Comb-drive finger model [8, 21].

Both models deals with the motion of one moveable capacitor plate under Coulomb forces and a DC-AC voltage $\hat{V}(t) = \hat{v}_0 + \hat{\delta}v(\omega t)$. Here $\hat{v}_0$ represents the DC-voltage and $\hat{\delta}v(\omega t)$ represents the AC-voltage, where $v(\omega t)$ is a $2\pi/\omega$–periodic function. For the Nathanson's model the moveable plate is attached to a linear spring and moves parallel to a stationary one in a media with positive viscosity coefficient (see figure 1). In appropriate units, the equation motion for the moveable capacitor is given by the following non-dimensional second-order-differential equation

$$\ddot{x} + x + a\dot{x} - \frac{b\hat{V}^2(t)}{(1-x)^2} = 0,$$

where $x \in (-\infty, 1)$ and the dot denotes derivative with respect to the time t. For the Comb-drive finger model the moveable plate (*finger*) is now located between two fixes ones (see figure 2). Again, in appropriate units, the transversal motions of the finger is ruled by the following non-dimensional second-order-differential equation

$$\ddot{x} + x + a\dot{x} - \frac{4b\hat{V}^2(t)x}{(1-x^2)^2} = 0,$$

where $x \in (-1, 1)$. In both models, a and b are positive parameters related to physically properties of the device, more precisely

$$a = \frac{\gamma}{\sqrt{km}}, \quad b = \frac{\epsilon_0 A}{2kd^3},$$

where γ is the damping coefficient, k is the stiffness of the linear spring, m is the mass of the moveable plate, ϵ_0 is the absolute dielectric constant of vacuum ($\epsilon_0 = 8.85 \times 10^{-12}$), A is the overlapping area between the plates, and d is initial gap between them.

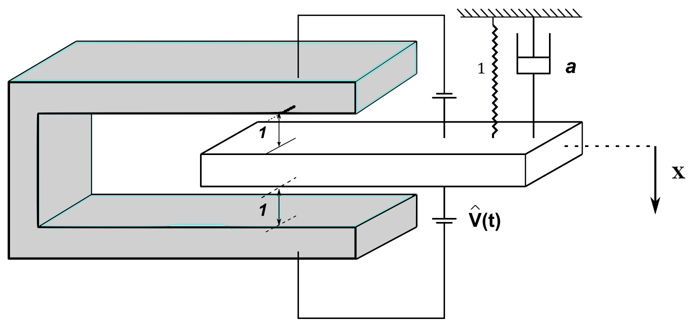

FIGURE 2. Idealized Comb-drive finger model with spring force $\mathbf{x}$, damping force $a\dot{\mathbf{x}}$, voltage source $\hat{V}(t)$ and initial gap between the moveable electrode and each fixed electrode is 1.

It is well know that for this kind of MEMS, there is the so called *pull-in* phenomenon. This emerges when the electrostatic force overcomes the resorting force in the device leading to its collapse (the electrodes collide).

For the Nathanson's model, when a DC voltage $\hat{v}_0$ is applied, the balance of the forces leading to the following relation between k, d, A and the applied voltage

$$\hat{v}_0^2 = \frac{8kd^3}{27\epsilon_0 A}, \tag{1}$$

in the literature, this DC voltage are known as *pull-in voltage* which we denoted by $v_{pull,N}$. On the other hand, for the Comb-drive finger model, the pull-in voltage $v_{pull,C}$ satisfies

$$v_{pull,C}^2 = \frac{27}{16} v_{pull,N}^2.$$

A straightforward computation shows that $b = \dfrac{4}{27 v_{pull,N}^2}$, in consequence the previous models can be written as

$$\ddot{x} + x + a\dot{x} - \frac{V^2(t)}{(1-x)^2} = 0,$$

and

$$\ddot{x} + x + a\dot{x} - \frac{\mathcal{V}^2(t)x}{(1-x^2)^2} = 0,$$

where $V(t) = \sqrt{\dfrac{4}{27}}\dfrac{\hat{V}(t)}{v_{pull,N}}$ and $\mathcal{V}(t) = \dfrac{\hat{V}(t)}{v_{pull,C}}$. More precisely,

$$V(t) = v_0 + \delta v(\omega t), \quad \text{where} \quad v_0 = \sqrt{\frac{4}{27}}\frac{\hat{v}_0}{v_{pull,N}}, \quad \delta = \sqrt{\frac{4}{27}}\frac{\hat{\delta}}{v_{pull,N}},$$

(2) and

$$\mathcal{V}(t) = \vartheta_0 + \lambda v(\omega t), \quad \text{where} \quad \vartheta_0 = \frac{\hat{v}_0}{v_{pull,C}}, \quad \lambda = \frac{\hat{\delta}}{v_{pull,C}},$$

Is clear from (1) that a small value of d produces small values of $v_{pull,N}$ too (also for $v_{pull,C}$). For instance, if we take reference state values for the Nathanson's model

$$A = 1.6 \times \times 10^{-9}\,[m^2], \quad a = 1.78 \times 10^{-6}\,[kg/s], \quad k = 0.17\,[N/m]$$

taken from [12], a gap $d = 3\times 10^{-6}\,m$ gives $v_{pull,N} = 9.8\,[V]$, and for $d = 3\times 10^{-8}\,m$ we obtain $v_{pull,N} = 0.0098\,[V]$. For qualitative and methodological purposes, we will assume a low viscosity regimen for both models by means of the following scaling $a \to \varepsilon$ with ε positive and small and a low voltage regimen for a Nathanson's model with a small d by means of the following scaling

$$v_0 \to \epsilon^{1/2} v_0, \quad \delta \to \epsilon^{1/2}\delta,$$

These assumptions allow to study the existence of periodic solutions of the following second–order non–autonomous differential equations

$$\ddot{x} + x + \varepsilon\left(\dot{x} - \frac{(v_0 + \delta v(\omega t))^2}{(1-x)^2}\right) = 0, \tag{3}$$

and

$$\ddot{x} + x + \varepsilon\dot{x} - \frac{(\vartheta_0 + \lambda v(\omega t))^2 x}{(1-x^2)^2} = 0, \tag{4}$$

where $v_0, \delta, \vartheta, \lambda$ are positive parameters defined in (2), $v(\omega t)$ is a $T = 2\pi/\omega$–periodic function. In particular for (3) we assume $\omega = q/p$ with q and p coprime positive integers and ε is a small parameter, also we shall provide information on the kind of stability of their periodic solutions.

In general to determine analytically the periodic solutions of a differential system is a very difficult work, many times impossible to do. But the structure of Nathanson model (3) suggests the use of the averaging theory as an alternative technique to find periodic solutions with non constant sign bifurcating from the periodic solutions of the harmonic oscillator $\ddot{x} + x = 0$. On the other hand, for (4) it is possible to prove under general conditions the locally asymptotically stability of the equilibrium $x \equiv 0$, which prevents the non existence of periodic solutions near to it. These are some reasons explaining why the averaging theory applied to the comb-drive finger model does not provide information on its periodic solution. So for studying the periodic solutions of the comb-drive finger model (4) we shall use the lower and upper solution approach. In section 2 we provide more information about the averaging theory that we shall use for studying the periodic solutions of the differential equation (3), and of the lower and upper solution method for the existence of periodic solution for the differential equation (4).

Our main results on the periodic solutions of the Nathanson's and comb-drive finger models are the following ones.

Theorem 1. *We define the functions*

$$h(t) = \frac{1}{2p\pi}\left(y_0\cos t - x_0 \sin t - \frac{\left(v_0 + \delta v\left(qt/p\right)\right)^2}{(1 - x_0\cos t - y_0 \sin t)^2}\right),$$

and

$$f_1(x_0,y_0) = -\int_0^{2p\pi} h(t)\sin t\,dt, \quad f_2(x_0,y_0) = \int_0^{2p\pi} h(t)\cos t\,dt.$$

Then for every $\varepsilon \neq 0$ sufficiently small, q and p coprime positive integers, and for every real (x_0^*, y_0^*) solution of the system

$$f_1(x_0, y_0) = 0, \qquad f_2(x_0, y_0) = 0, \tag{5}$$

satisfying

$$\det\left(\left.\frac{\partial(f_1, f_2)}{\partial(x_0, y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)} \right) \neq 0, \tag{6}$$

the Nathanson's model (3) *has a $2p\pi$–periodic solution $x(t,\varepsilon)$ which tends to the periodic solution*

$$x(t) = x_0^* \cos t + y_0^* \sin t, \tag{7}$$

of the differential equation $\ddot{x} + x = 0$ when $\varepsilon \to 0$ traveled p times. Moreover, if the real part of all the eigenvalues of the matrix

$$\left(\left.\frac{\partial(f_1, f_2)}{\partial(x_0, y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)} \right) \tag{8}$$

are negative the periodic solution $x(t,\varepsilon)$ is stable, and if some of those real parts is positive the periodic solution $x(t,\varepsilon)$ is unstable.

Theorem 1 is proved using the averaging theory in section 3.

In what follows we present the following corollary of Theorem 1.

Corollary 1. *Under the assumptions of Theorem* 1 *consider the Nathanson's model* (3) *with $p = q = 1$ and $v(t) = \sin t$. Then for $\varepsilon \neq 0$ sufficiently small the differential equation* (3) *has a 2π–periodic solution $x(t,\varepsilon)$ tending to the periodic solution* (7) *of the differential equation $\ddot{x} + x = 0$ when $\varepsilon \to 0$, for each real zero (x_0^*, y_0^*) of the system*

(9)

$$\begin{aligned}
f_1(x_0,y_0) = \quad & \frac{1}{2\,|x_0+1|\left(1-x_0^2-y_0^2\right)^{3/2}\left(x_0^2+y_0^2\right)^3} \Bigg(|x_0+1| \left(1-x_0^2-y_0^2\right)^{3/2} \\
& \Big(a x_0 \left(x_0^2+y_0^2\right)^3 + 4b\delta \left(v_0\left(y_0^4-x_0^4\right) + \delta y_0 \left(y_0^2-3x_0^2\right)\right)\Big) \\
& +2b(x_0+1)\big(v_0^2 y_0 \left(x_0^2+y_0^2\right)^3 - 2\delta v_0 \left(x_0^6-x_0^4-3x_0^2y_0^4-2y_0^6+y_0^4\right) \\
& +\delta^2 y_0 \left(3x_0^6+x_0^4\left(6y_0^2-9\right)+3x_0^2\left(y_0^4-2y_0^2+2\right)+3y_0^4-2y_0^2\right)\big)\Bigg), \\
f_2(x_0,y_0) = \quad & -\frac{1}{2\left(1-x_0^2-y_0^2\right)^3\left(x_0^2+y_0^2\right)^3} \Bigg(\left(1-x_0^2-y_0^2\right)^{3/2} \\
& \big(2b\delta x_0 \big(2v_0y_0\left(x_0^2+y_0^2\right)\left(3x_0^2+3y_0^2-2\right) + \delta\big(x_0^6-3x_0^4 \\
& +x_0^2\left(-3y_0^4+6y_0^2+2\right)-2y_0^6+9y_0^4-6y_0^2\big)\big) - a y_0 \left(1-x_0^2-y_0^2\right)^{3/2} \\
& \left(x_0^2+y_0^2\right)^3\big) + 2bx_0\left(1-x_0^2-y_0^2\right)\big(v_0^2\sqrt{1-x_0^2-y_0^2}\left(x_0^2+y_0^2\right)^3 \\
& +4\delta v_0 y_0 \left(1-x_0^2-y_0^2\right)^2\left(x_0^2+y_0^2\right) - 2\delta^2\left(x_0^2-3y_0^2\right)\left(1-x_0^2-y_0^2\right)^2\big)\Bigg),
\end{aligned}$$

satisfying (6) *the Nathanson's model* (3) *has a 2π–periodic solution $x(t,\varepsilon)$ which tends to the periodic solution* (7) *of the differential equation $\ddot{x} + x = 0$ when $\varepsilon \to 0$. Moreover, if the real part of all the eigenvalues of the matrix* (8) *corresponding to the previous functions f_1 and f_2 are negative the periodic solution $x(t,\varepsilon)$ is stable, and if some of those real parts is positive the periodic solution $x(t,\varepsilon)$ is unstable.*

Corollary 1 is proved at the end of section 3.

Under the assumptions of Corollary 1 and taking the values

$$v_0 = \frac{1}{2}\sqrt{\frac{4}{27}}, \qquad \delta = \frac{1}{4}\sqrt{\frac{4}{27}},$$

the system (9) only provides a unique periodic solution associated to its zero

$$(x_0^*, y_0^*) = (-0.0368200491.., -0.0028977591..),$$

for which the determinant (6) is 0.2517304269.., and the eigenvalues of the matrix (8) are $0.5 \pm 0.0415984005..i$. So the corresponding periodic solution is locally stable.

We define

$$\mathcal{V}_M := \max_{t\in[0,T]} \mathcal{V}(t), \quad \mathcal{V}_m := \min_{t\in[0,T]} \mathcal{V}(t), \quad \lambda := \frac{\mathcal{V}_m}{\mathcal{V}_M},$$

and when $0 < \mathcal{V}_M < 1$,

$$\lambda_c := \sqrt{\frac{\mathcal{V}_M}{4 - 3\mathcal{V}_M}}.$$

Under these definitions we are able to present our results on the comb-drive model.

Theorem 2. *Assume the conditions*

(i) $0 < \mathcal{V}_M < 1$.
(ii) $0 \le \varepsilon < 2\sqrt{1 - \mathcal{V}_M^2}$.
(iii) $T < 2/\sqrt{1 - (\mathcal{V}_m^2 + \frac{\varepsilon^2}{4})}$.

Then the equilibrium $x \equiv 0$ of the comb-drive model (4) is locally asymptotically stable.

Theorem 3. *Assume that $0 < \mathcal{V}_M < 1$ then there exists a positive T-periodic solution $\varphi(t)$ of the comb-drive model (4) such that*

$$\sqrt{1 - \mathcal{V}_M} < \varphi(t) < \sqrt{1 - \mathcal{V}_m},$$

for all $t \in \mathbb{R}$. Moreover, if $\lambda > \lambda_c$ and ε is small enough, then the periodic solution $\varphi(t)$ is unstable.

Remark 1. *Due to the odd symmetry of the differential equation (4) the Theorem 3 implies the existence of a negative T-periodic solution $-\varphi(t)$ which is also unstable.*

Combining the previous results we have:

Corollary 2. *Under the assumptions of Theorem 2 for $\varepsilon = 0$ and $\lambda > \lambda_c$ the comb-drive model (4) has the equilibrium $x \equiv 0$ locally asymptotically stable for ε small enough. Moreover, it has a couple of T-periodic solutions with defined sign which are unstable.*

Theorem 2 and Theorem 3 are proved using the Floquet's theory and the lower and upper solution method of section 3.

2. Some methods for computing periodic solutions

In this section we present the basic results from the averaging theory and the lower and upper solution technique that we need for proving the results stated in section 1.

2.1. Averaging.

Consider the differential system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{10}$$

where Ω is an open subset of $\mathbb{R}^n$ and $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is a $\mathcal{C}^2$ function T–periodic in the first variable. Assume that system (5) has a submanifold of dimension n of T–periodic solutions contained in Ω.

We study which T–periodic solutions of the differential system (4) persist as periodic solutions of the perturbed differential system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{11}$$

where $\varepsilon \neq 0$ is sufficiently small, and the functions $F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions T–periodic in the first variable. As we shall see in what follows under convenient assumptions the averaging theory provides a solution of this problem.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the differential system (11) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. Suppose that $\mathbf{x}(t, \mathbf{z}, 0)$ is a periodic solution of the unperturbed differential system (10). We consider the first variational differential equations of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$, i.e.

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}, \tag{12}$$

where $\mathbf{y}$ is an $n \times n$ matrix. Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of the linear differential system (12) such that $M_{\mathbf{z}}(0)$ is the $n \times n$ identity matrix.

We have assumed that there is an open set V such that $\mathrm{Cl}(V) \subset \Omega$ and for each $\mathbf{z} \in \mathrm{Cl}(V)$ the orbit $\mathbf{x}(t, \mathbf{z}, 0)$ is T–periodic. Note that the set $\mathrm{Cl}(V)$ is *isochronous* for the differential system (10), because it is formed only by T–periodic orbits. Therefore we can provide an answer to the problem of the persistence for ε sufficiently small of T–periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $\mathrm{Cl}(V)$ as follows.

The next result is proved in [3], but it was already stated in Malkin [10] and Roseau [16], see also [18, 20].

Theorem 4 (Perturbations of an isochronous set of periodic solutions)**.** *Suppose that there is an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that the solution $\mathbf{x}(t, \mathbf{z}, 0)$ for all $\mathbf{z} \in \mathrm{Cl}(V)$ is T–periodic. Define the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T} \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}, 0)) dt. \tag{13}$$

If there is $\mathbf{a} \in V$ with $\mathcal{F}(\mathbf{a}) = 0$ and $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(\mathbf{a})\right) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (11) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{a}$ as $\varepsilon \to 0$. Moreover, if the real part of all the eigenvalues of the matrix $\left(d\mathcal{F}/d\mathbf{z}\right)(\mathbf{a})$ are negative, the periodic solution $\mathbf{x}(t, \varepsilon)$ is stable, and if some of those real parts is positive the periodic solution $\mathbf{x}(t, \varepsilon)$ is unstable.

We note that Theorem 4 detects periodic solutions of the differential system (10) which can be continued to periodic solutions of the differential system (11).

2.2. **Lower and upper solutions.** Now we present a brief introduction to the notion of lower and upper solution for the second–order differential equations. For more information on this method, the definitions and the theorems presented in this subsection see [4].

Consider a second–order differential equation of the form

$$\ddot{x} = g(t, x, \dot{x}), \tag{14}$$

where $g : D \to \mathbb{R}$, $D = \mathbb{R} \times (a, b) \times \mathbb{R}$ is a continuous function $-\infty \le a < b \le \infty$, $\partial_x g$ is continuous on D and

$$g(t + T, x, \dot{x}) = g(t, x, \dot{x}) \quad \forall (t, x, \dot{x}) \in D,$$

where $T > 0$ the period of the function g with respect to the variable t.

The problem of finding T-periodic solutions $x(t)$ of the second–order differential equation (14) with $\dot{x}(0) = \dot{x}(T) = 0$ is equivalent to solve the following periodic boundary problem

$$\ddot{x} = g(t, x, \dot{x}), \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T) = 0. \tag{15}$$

A function $\alpha \in C^2((0, T)) \cap C^1([0, T])$ is a *lower*–solution of (15) relative to the domain D if

(i) For all $t \in [0, T]$, $\alpha(t) \in (a, b)$ and $\ddot{\alpha}(t) \ge g(t, \alpha(t), \dot{\alpha}(t))$.
(ii) $\alpha(0) = \alpha(T)$, $\quad \dot{\alpha}(0) \ge \dot{\alpha}(T)$.

A function $\beta \in C^2((0, T)) \cap C^1([0, T])$ is *upper*–solution of (15) relative to the domain D if the inequalities of the previous definition hold in the reversed order.

Define

$$E := \{(t, x, v) \in D \,|\, \alpha(t) \le x \le \beta(t)\}, \tag{16}$$

and suppose that g satisfies

$$|g(t, x, v)| \le \psi(|v|), \quad \forall (t, x, v) \in E, \tag{17}$$

where $\psi : [0, +\infty) \to (0, +\infty)$ is some positive continuous function such that

$$\int_0^\infty \frac{m}{\psi(m)} dm = \infty. \tag{18}$$

The condition (18) is known as the *Nagumo condition* for the function g on E. Now we cite a classical result which allows to obtain periodic solutions using lower and upper solutions.

Theorem 5. *Let $\alpha, \beta \in C^2([0, T])$ be a of lower– and upper–solution of the boundary problem (15) relative to the domain D respectively, such that $\alpha(t) \le \beta(t)$ for all $t \in [0, T]$, E be defined as in (16), $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a positive continuous function satisfying (18), and $g : E \to \mathbb{R}$ be a continuous function which satisfies (17). Then the boundary problem (15) has at least one solution $\varphi \in C^2([0, T])$ such that $\alpha(t) \le \varphi(t) \le \beta(t)$ for all $t \in [0, T]$.*

3. Proof of Theorem 1

We write the second–order differential equation (3) as the differential system of first order

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -x - \varepsilon\left(\dot{x} - \frac{(v_0+\delta v(\omega t))^2}{(1-x)^2}\right).
\end{aligned} \tag{19}
$$

This differential system is written into the normal form (11) for applying to it the averaging theory described in section 2, where using the notation of that section we have

$$
\begin{aligned}
\mathbf{x} &= (x,y), \\
F_0(t,\mathbf{x}) &= (y,-x), \\
F_1(t,\mathbf{x}) &= \left(0, \dot{x} - \frac{(v_0+\delta v(\omega t))^2}{(1-x)^2}\right), \\
F_2(t,\mathbf{x}) &= (0,0), \\
\mathbf{z} &= (x_0,y_0).
\end{aligned}
$$

Solving the unperturbed differential system (10) corresponding to system (19), we obtain that it has the 2π–periodic solutions $(x(t,\mathbf{z}),y(t,\mathbf{z}))$ given by

$$
x(t,x_0,y_0) = x_0\cos t + y_0 \sin t, \qquad y(t,x_0,y_0) = y_0\cos t - x_0\sin t.
$$

Now we compute the fundamental matrix $M_{\mathbf{z}}(t)$ associated to the variational differential system (12) corresponding to the unperturbed differential system (19). Thus an easy computation provides

$$
M_{\mathbf{z}}(t) = M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.
$$

We compute the averaged function (13), i.e.

$$
\begin{aligned}
\mathcal{F}(x_0,y_0) &= \frac{1}{2p\pi}\int_0^{2p\pi} M^{-1}(t)F_1(t,x(t,x_0,y_0),y(t,x_0,y_0))dt \\
&= \left(f_1(x_0,y_0), f_2(x_0,y_0)\right),
\end{aligned}
$$

where these last two functions are the ones defined in the statement of Theorem 1. Therefore, by Theorem 3 it follows that for every $\varepsilon \neq 0$ sufficiently small and for every (x_0^*,y_0^*) solution of system (5) satisfying (6), the non–autonomous differential equation (3) has a $2p\pi$–periodic solution $x(t,\varepsilon)$ which tends to the $2p\pi$–periodic solution (7) of the differential equation $\ddot{x}+x=0$ when $\varepsilon\to 0$ traveled p times. Moreover, if the real part of all the eigenvalues of the matrix (8) are negative the periodic solution $x(t,\varepsilon)$ is stable, and if some of those real parts is positive the periodic solution $x(t,\varepsilon)$ is unstable. This completes the proof of Theorem 1.

Proof of Corollary 1. Take $q=p=1$. Then we compute the two functions $f_1(x_0,y_0)$ and $f_2(x_0,y_0)$ defined in the statement of Theorem 1 and we get the functions (9). Hence, by Theorem 1 the proof of the corollary follows. □

4. Proof of Theorems 2 and 3

For the proofs of the Theorems 2 and 3 we need the following preliminary results.

Lemma 6. *Consider a differential equation of the form*

$$\ddot{u} + q_\varepsilon(t, u, \dot{u}) = 0, \tag{20}$$

where $q_\varepsilon(t, u, \dot{u}) = \varepsilon\dot{u} + q(t)u$ *and* $\varepsilon \in \mathbb{R}^+, q \in C(\mathbb{R}/T\mathbb{Z})$. *The change of variable*

$$y = e^{\varepsilon t/2}u, \tag{21}$$

transforms the equation (20) *into the Hill's equation*

$$\ddot{y} + Q_\varepsilon(t)y = 0, \tag{22}$$

with $Q_\varepsilon(t) = q(t) - \dfrac{1}{4}\varepsilon^2$. *Moreover, if* λ_1, λ_2 *are the Floquet's multipliers of equation* (20) *and* μ_1, μ_2 *are the Floquet's multipliers of equation* (22), *then*

$$\lambda_i = e^{-\varepsilon T/2}\mu_i, \quad i = 1, 2.$$

Proof. The first claim is a well known fact and can be consulted in [11] page 51. For the second part consider y_1, y_2 the Floquet's solutions of (22) with multipliers λ_1, λ_2 respectively. Define

$$u_i(t) = e^{-\varepsilon t/2}y_i(t), \quad i = 1, 2.$$

Then u_1, u_2 are solutions of (20) and moreover are Floquet solutions. In fact,

$$\begin{aligned} u_i(t+T) &= e^{-\varepsilon(t+T)/2}y_i(t+T), \\ &= e^{-\varepsilon T/2}\mu_i e^{-\varepsilon t/2}y_i(t) \\ &= \lambda_i u_i(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

□

For the following lemma we need some preliminary definitions and results. Let $a(t) \in C(\mathbb{R}/T\mathbb{Z})$ and consider the Hill's equation

$$\ddot{y} + a(t)y = 0, \tag{23}$$

with the corresponding Floquet multipliers $\rho_i = \rho_i[a], i = 1, 2$. Equation (23) is *stable* if for any solution $y(t)$ there exists a positive constant M such that

$$\sup_t \{|y(t)| + |\dot{y}(t)|\} < M,$$

otherwise, we say that equation (23) is *unstable*. Moreover (22) is stable if and only the Floquet's multipliers satisfy some of the following conditions

- (I) $\rho_1 = \overline{\rho_2} \notin \mathbb{R}$, $|\rho_{1,2}| = 1$,
- (II) $\rho_{1,2} = \pm 1$ and the monodromy matrix is equal to $\pm I_d$ being I_d the identity matrix,

and is *unstable* if

- (III) $\rho_{1,2} = \pm 1$ and the monodromy matrix is not equal to $\pm I_d$.
- (IV) $|\rho_1| < 1 < |\rho_2|$, $\rho_i \in \mathbb{R}$, $i = 1, 2$.

Equation (23) is called

- *Elliptic* if condition (I) holds.
- *Parabolic Stable* if the condition (II) holds.
- *Parabolic Unstable* if the condition (III) holds.
- *Hyperbolic* if the condition (IV) holds.

Lemma 7. *For the differential equation* (20) *the following statements hold.*

(a) *If* (22) *is hyperbolic for* $\varepsilon = 0$ *then the solution* $u \equiv 0$ *of* (20) *is hyperbolic for* ε *small enough.*

(b) *If* (22) *is elliptic then solution* $u \equiv 0$ *of* (20) *is locally asymptotically stable for* ε *small enough.*

(c) *If* (22) *is elliptic for* $\varepsilon = 0$ *then the solution* $u \equiv 0$ *of* (20) *is locally asymptotically stable for* ε *small enough.*

Proof. (a) It is easy to check that the Floquet's multipliers $\mu_{1,0}, \mu_{2,0}$ of equation (22) for $\varepsilon = 0$ verify $|\mu_{1,0}| < 1 < |\mu_{2,0}|$. Furthermore, the Floquet's multipliers $\mu_1(\varepsilon), \mu_2(\varepsilon)$ are continuous functions and satisfy that $\mu_1(0) = \mu_{1,0}$, $\mu_2(0) = \mu_{2,0}$. Thus equation (22) is hyperbolic for ε small enough and we get $|\mu_1(\varepsilon)| < 1 < |\mu_2(\varepsilon)|$ for $\varepsilon \ll 1$. By Lemma 6 the Floquet's multipliers λ_1, λ_2 satisfy $\lambda_i(e) = e^{-\varepsilon T/2}\mu_i(e), \quad i = 1, 2$. Therefore we deduce that $|\lambda_1(e)| < 1$ and $|\lambda_2(e)| = e^{-\varepsilon T/2}$ $\mu_2(e) > 1$ for $\varepsilon \ll 1$ because $|\lambda_2(0)| = |\mu_{2,0}| > 1$.

(b) If equation (22) is elliptic then its solutions are bounded, i.e. for each solutions $y(t)$ there exits a positive constant M such that $|y(t)| + |\dot{y}(t)| \leq M, \quad \forall t \in \mathbb{R}$. From Lemma 6 the corresponding solutions $u(t) = e^{-\varepsilon t/2}y(t)$ of equation (20) are convergent to zero when $t \to \infty$, i.e. $|u(t)| + |\dot{u}(t)| \to 0$, when $t \to \infty$. The stability of $u \equiv 0$ is an easy consequence of the stability of $y \equiv 0$ for (22) using the relation $u = e^{-\varepsilon t/2}y$.

(c) This statement follows directly from statement (b) because the ellipticity implies the strong ellipticity, i.e. the Hill's equation (22) remains elliptic for small values of ε. □

Proof of Theorem 2. The variational equation at $x = 0$ of equation (4) is given by

$$\ddot{u} + q_\varepsilon(t, u, \dot{u}) = 0,$$

with $q_\varepsilon(t, u, \dot{u}) = \varepsilon\dot{u} + (1 - \mathcal{V}^2(t))u$. The change of variable (21) transform this equation into the Hill's equation (22) with $Q_\varepsilon(t) = 1 - (\mathcal{V}^2(t) + \frac{1}{4}\varepsilon^2)$. From the hypothesis (ii) and (iii) we have

$$0 < Q_\varepsilon(t) < 1 - (\mathcal{V}_m + \frac{1}{4}\varepsilon^2) < \frac{4}{T^2}.$$

Therefore $T\displaystyle\int_0^T Q_\varepsilon(t)dt < 4$. From the classical Lyapunov-Borg criterion (see [11]) the ellipticity of (22) follows. A direct application of Lemma 7 statement (b) completes the proof. □

Proof of Theorem 3. A straightforward computation shows that the roots in $(0, 1)$ of the equations

$$(1 - x^2)^2 = \mathcal{V}_M^2, \quad (1 - x^2)^2 = \mathcal{V}_m^2,$$

are constant lower– and upper–solutions respectively for equation (4). Thus $x_L := \sqrt{1-\mathcal{V}_M}$ is a lower–solution and $x_U := \sqrt{1-\mathcal{V}_m}$ is an upper–solution of (4) that verify $0 < x_L < x_U < 1$. Notice that the function $g(t,x,v) = -\varepsilon v - x + \dfrac{\mathcal{V}^2 x}{(1-x^2)^2}$ satisfies the hypothesis of Theorem 5 on the region $D = \mathbb{R} \times (0,1) \times \mathbb{R}$ with $E := \{(t,x,v) \in D \,|\, x_L \le x \le x_U\}$. The Nagumo condition (18) is satisfied because

$$|g(t,x,v)| \le \varepsilon |v| + C, \quad \text{with} \quad C = x_U + \frac{\mathcal{V}_M x_U}{(1-x_U^2)^2}$$

and

$$\int_0^\infty \frac{m}{\varepsilon m + C} dm = \infty.$$

Hence, by Theorem 5 it follows that there exists a T-periodic solution $\varphi(t)$ of equation (4) such that

$$x_L \le \varphi(t) \le x_U, \tag{24}$$

for all $t \in \mathbb{R}$. In order to study the stability properties of the solution $\varphi(t)$ we consider the associated variational equation

$$\ddot{u} + \varepsilon \dot{u} + \hat{q}(t) u = 0, \quad \text{with} \quad \hat{q}(t) = 1 - \frac{(1+3\varphi^2(t))\mathcal{V}^2(t)}{(1-\varphi^2(t))^3}.$$

Using (24) and doing some elementary computations we obtain

$$\hat{q}_* < \hat{q}(t) < \hat{q}^*,$$

where

$$\hat{q}_* := 1 - \frac{(4-3\mathcal{V}_m)\mathcal{V}_M^2}{\mathcal{V}_m^3}, \quad \hat{q}^* := 1 - \frac{(4-3\mathcal{V}_M)\mathcal{V}_m^2}{\mathcal{V}_M^3}.$$

We claim that $\hat{q}_* < 0$ if $\lambda > \lambda_c$. Indeed,

$$\hat{q}_* < 0 \Leftrightarrow \mathcal{V}_M^3 - 4\mathcal{V}_m^2 + 3\mathcal{V}_M \mathcal{V}_m^2 < 0 \Leftrightarrow \frac{\mathcal{V}_M}{4-3\mathcal{V}_M} < \left(\frac{\mathcal{V}_m}{\mathcal{V}_M}\right)^2 \Leftrightarrow \lambda_c < \lambda.$$

Notice that if $\varepsilon = 0$ we have the hyperbolic Hill's equation

$$\ddot{u} + \hat{q}(t) u = 0 \tag{25}$$

because $\hat{q}(t) < \hat{q}_* < 0$ for all $t \in \mathbb{R}$. Applying statement (a) of Lemma 7 we obtain that $\varphi(t)$ is hyperbolic and therefore unstable. □

Concluding Remarks

In Theorem 1 we provide the functions whose simple zeros allow to compute the periodic solutions of the Nathanson's model and also how to study the stability of these periodic solutions. As an application of our results for the function $v(\omega t) = \sin t$, we have computed explicitly the function whose zeros provide the periodic solutions of the Nathanson's model, see Corollary 1.

In the comb-drive finger model the generic character of attraction of the origin prevents the non–appearance of oscillatory periodic solutions. Thus, using the lower and upper solution techniques and before reaching the pull in voltage, we have been able to capture two periodic solutions $\pm\varphi(t)$ with period equal to the AC–voltage each one having constant sign. These periodic solutions are generically unstable for the viscosity coefficient ε low enough. The generic condition is related to a critical value λ_c for the ratio $\lambda = V_m/V_M$, namely $\lambda > \lambda_c$. Moreover, the equilibrium zero is a local attractor for ε small enough and coexists with the periodic solutions $\pm\varphi(t)$ for ε small enough and high frequencies for the AC–voltage. This makes that the MEMS devices a nonlinear oscillator that usually stabilizes at the origin. The following question remains open. What happens with the stability of the periodic solution $\varphi(t)$ if $\lambda \leq \lambda_c$? The variational equation (25) with $\varepsilon = 0$ is still hyperbolic below the critical value λ_c?. The difficulty here is that it is not possible to apply directly the well known criteria of stability for Hill's equations. Can be the zero equilibrium an attractor for higher values of ε and lower frequencies?

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