ON THE INTEGRABILITY OF A THREE–DIMENSIONAL FORCED–DAMPED DIFFERENTIAL SYSTEM

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Abstract. In 2011 Pehlivan proposed a three–dimensional forced–damped autonomous differential system which can display simultaneously unbounded and chaotic solutions. This untypical phenomenon has been studied recently by several authors. In this paper we study the opposite to its chaotic motion, i.e. its integrability, mainly the existence of polynomial and rational first integrals through the analysis of its invariant algebraic surfaces.

1. Introduction and statement of the main result

We consider in $\mathbb{R}^3$ the autonomous system of differential equations

$$\begin{aligned} \dot{x} &= -ax + y + yz, \\ \dot{y} &= x - ay + bxz, \\ \dot{z} &= cz - bxy, \end{aligned} \tag{1}$$

where a, b, c are real parameters. This system arise in mechanical, electrical and fluid–dynamical contexts, see for more details the articles of Miyaji, Okamoto and Craik [6, 7] and the references quoted there. This system was proposed and studied by Pehlivan [8]. The system extends a previous study of Craik and Okamoto [1], including linear forcing and damping.

Pehlivan showed that system (1) displays simultaneously unbounded and chaotic solutions. This phenomenon has been studied in more depth by Miyaji, Okamoto and Craik who also find that can be accompanied by three distinct period–doubling cascades of periodic orbits to chaos.

Chaotic systems are nonlinear deterministic systems which exhibits a complex and unpredictable behavior, hence it is a very interesting phenomenon in nonlinear dynamical systems and it has been intensively studied starting with the Lorenz system. The majority of the known chaotic system have one or more quadratic non-linearities. The existence of quadratic nonlinearities may increase the chaoticity of the system, so in this paper we do not consider the case $b = 0$.

As far as we know this rich dynamical system (1) has never been investigated from the integrability point of view. The main goal of this paper is to characterize the polynomial and rational first integrals of system (1). For doing this we need to provide a complete characterization of the invariant algebraic surfaces of system (1) depending on its parameters. In order to obtain such invariant algebraic surfaces we shall use the Darboux theory integrability which gives a link between the algebraic geometry of the system and its first integrals, see for more details about this theory [2, 3, 4, 5].

It is well known that the existence of a first integral for three–differential system allows to reduce the study of its dynamics in one dimension, and that the existence of two independent first integrals allows to describe completely the dynamics of the system.

Let U be an open and dense subset of $\mathbb{R}^3$. A nonconstant function $H\colon U \to \mathbb{R}$ is called a *first integral* of system (1) on U if $H(x(t), y(t), z(t))$ is constant for all of the values of t for which $(x(t), y(t), z(t))$ is a solution of system (1) contained in U. So H is a first integral of system (1) if and only if

$$(-ax + y + yz)\frac{\partial H}{\partial x} + (x - ay + bxz)\frac{\partial H}{\partial y} + (cz - bxy)\frac{\partial H}{\partial z} = 0,$$

for all $(x, y, z) \in U$. If H is a polynomial (respectively a rational function) we say that H is a *polynomial* (respectively *rational) first integral*.

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