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Counterexample to a Conjecture on the Algebraic Limit Cycles of Polynomial Vector Fields

JAUME LLIBRE and CHARA PANTAZI
Departament de Matemátiques, Universitat Autónoma de Barcelona, 08193 Bellaterra, Barcelona, Spain. e-mail: jllibre, hpant@mat.uab.es

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Abstract. In Geometriae Dedicata **79** (2000), 101–108, Rudolf Winkel conjectured: for a given algebraic curve $f = 0$ of degree $m \geqslant 4$ there is in general no polynomial vector field of degree less than $2m - 1$ leaving invariant $f = 0$ and having exactly the ovals of $f = 0$ as limit cycles. Here we show that this conjecture is not true.

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1. Introduction

A planar vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{1}$$

is *a polynomial of degree n* if P and Q are real polynomials in the variables x and y, and the maximum degree of P and Q is n.

A periodic orbit of a vector field X in $\mathbb{R}^2$ is a *limit cycle* if it is isolated in the set of all periodic orbits of X.

In 1900, Hilbert [4] in the second part of his 16th problem, proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane. This has been one of the main problems in the qualitative theory of planar differential equations in the 20th century. The contributions of Écalle [2] and Ilyashenko [5] proving that any polynomial vector field has finitely many limit cycles have been the best results in this area. But until now it has not been proved that a uniform upper bound exists. This problem remains open even for the quadratic polynomial vector fields.

If $f = f(x, y)$ is an irreducible polynomial of degree m in the ring $\mathbb{R}[x, y]$, then $f = 0$ is an *irreducible algebraic curve of degree m* in $\mathbb{R}^2$. A limit cycle is *algebraic* of *degree m* if it is contained in an irreducible algebraic curve of degree m.

Hilbert also asked about the possible distributions of the limit cycles of polynomial vector fields. Recently, it has been proved that any finite configuration of limit cycles is realizable by polynomial vector fields. More precisely, we say that a *con-*