ON THE EXTENSION OF SHARKOVSKIĬ'S THEOREM TO CONNECTED GRAPHS WITH NON-POSITIVE EULER CHARACTERISTIC

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This paper contains a characterization of all possible sets of periods for all continuous self-maps on a connected topological graph with zero Euler characteristic having all branching points fixed. A characterization in terms of linear orderings is given for the simplest connected topological graph with zero Euler characteristic that has a branching point, the topological graph shaped like the letter σ. In this case a proof follows by lifting the continuous self-map on σ. We show the difficulties that arise in the simplest connected topological graphs with negative Euler characteristic, like for instance the topological graph shaped like the figure 8.

1. Introduction and Statement of the Results

A *finite graph* (simply a *graph*) G is a Hausdorff space which has a finite subspace V (points of V are called *vertices*) such that $G\backslash V$ is a disjoint union of finite number of open subsets $e_1, e_2, \ldots, e_k$ (called *edges*), each e_i is homeomorphic to an open interval of the real line, and one or two vertices are attached at the boundary of each edge.

Note that a graph is compact, since it is a union of finite number of compact subsets (the closed edges $\overline{e}_i$ and the vertices). It may be connected or disconnected, and it may have isolated vertices.

The number of edges having a vertex as an end point (with the closed edges homeomorphic to a circle counted twice) will be called the *valence* of this vertex. A vertex of valence 1 is called an *end point* and a vertex of valence ≥ 3 is called a *branching point*. Given a graph G, $e(G)$ and $b(G)$ will denote the number of its end points and branching points respectively.

The rational homology groups of G are well-known: $H_0(G; \mathbb{Q}) \approx \mathbb{Q}^c$ and $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^d$, where c and d are the number of connected components of G and the number of independent circuits of G, respectively. A *circuit* is a subset of G homeomorphic to a circle. The *Euler characteristic* $\chi(G)$ *of* G is $c - d$. We will call a continuous self-map f of G having all branching points of G as fixed points a *G-map*. A point x of G will be called *periodic* with respect to f of *period* n if n is the smallest positive integer such that $f^n(x) = x$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f and by $\mathbb{N}$ the set of positive integers.