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Limit cycles for a class of second order differential equations

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ABSTRACT

We study the limit cycles of a wide class of second order differential equations, which can be seen as a particular perturbation of the harmonic oscillator. In particular, by choosing adequately the perturbed function we show, using the averaging theory, that it is possible to obtain as many limit cycles as we want.

1. Introduction and statement of the main results

The second order differential equations arise in many areas of science and technology essentially, but in the last years many important applications in social science, economics and business administration have raised, as well as plenty of applications in physics, chemistry, biology and the engineries. Nevertheless, in general it is impossible to determine the solution of a second order differential equation in terms of explicit functions, so we look for properties of these kind of equations which can be obtained without solving them. The problem is more relevant in perturbation theory, where the periodic orbits play a main role. For more information about the study of the periodic orbits of second order differential equations see for instance the references [6–8,13,14], and the ones quoted there.

It is well known that all solutions of the harmonic oscillator are periodic with the same period, i.e. the origin of this system is an isochronous center. When we take a perturbation of the harmonic oscillator, which periodic orbits persist? Which periodic orbits generate limit cycles? In this Letter we ask these questions for a wide class of second order differential equations, which are essentially a perturbation of the harmonic oscillator. More precisely, we study equations of the form

$$\ddot{x} = -\left(1 + \varepsilon\kappa \cos^2 t\right)x + \varepsilon f(t, x, \dot{x}), \tag{1}$$

where ε is a small parameter, κ is a real parameter and the function $f(t, x, \dot{x})$ is at least of class C^2 and 2π-periodic in the variable t in order to apply the averaging theory [9].

We would like to mention the Ermakov systems, which have a long history in sciences and many important applications in physics. In the last years many people have retook their study, see for instance [2,4,5]. Since the classical Ermakov system has the form

$$\ddot{x} = -\omega^2(t)x + \frac{1}{x^3}G(x).$$

The class of second order differential equations (1) contains the subclas of Ermakov systems given by the functions $G(x) = \varepsilon x^3 f(x)$ and $\omega^2(t) = 1 + \varepsilon\kappa \cos^2 t$.

The first goal of this Letter is to provide a general result which, for ε sufficiently small, allows us to study the periodic orbits of the second differential equation (1) for an arbitrary function $f(t, x, \dot{x})$, see the next Theorem 1. After, we shall compute explicitly the periodic orbits of such second order differential equations for some explicit functions $f(t, x, \dot{x})$, see the next propositions (Propositions 2 and 3).

For stating our results we need some definitions. Thus we define the functions

$$\begin{aligned} h_1(s, r, a) &= \kappa r \cos s \cos^2(a + s)\sin(a + s) \\ &\quad - f(a + s, r\cos s, -r\sin s)\sin(a + s), \\ h_2(s, r, a) &= -\kappa r \cos s \cos^3(a + s) \\ &\quad + f(a + s, r\cos s, -r\sin s)\cos(a + s), \end{aligned}$$

and

$$\mathrm{f}_j(r, a) = \frac{1}{2\pi}\int_0^{2\pi} h_j(s, r, a)\, ds \quad \text{for } j = 1, 2.$$

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