A Note on the First Integrals of Vector Fields with Integrating Factors and Normalizers

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Abstract. We prove a sufficient condition for the existence of explicit first integrals for vector fields which admit an integrating factor. This theorem recovers and extends previous results in the literature on the integrability of vector fields which are volume preserving and possess nontrivial normalizers. Our approach is geometric and coordinate-free and hence it works on any smooth orientable manifold.

Key words: first integral; vector field; integrating factor; normalizer

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1 Introduction and statement of the main result

Let M be a smooth orientable manifold of dimension n endowed with a volume form Ω. We denote by $C^\infty(M)$ and $\mathfrak{X}^\infty(M)$ the spaces of smooth real-valued functions and vector fields on M respectively. We shall only work in the smooth (C^∞) category, although most of our results can be easily adapted to lower regularity.

This letter is focused on the integrability properties of vector fields. More precisely, we are interested in proving some sufficient conditions which imply the existence of explicit first integrals (i.e. computable). We recall that a function $H \in C^\infty(U)$, where $U \subseteq M$ is an open subset of M, is a *first integral* of the vector field X in U if $L_X(H) = 0$, which implies that H is constant along the integral curves of X. As usual, L_X denotes the Lie derivative with respect to the vector field X. We recall that an equivalent way of writing the first integral condition is $L_X(H) = i_X(dH) = X(H) = 0$, where d is the exterior derivative and i denotes de contraction operator. In this paper we are interested in global or semi-global first integrals in the sense that the open set $U \subseteq M$ where the first integral is well defined is known a priori.

The importance of the existence of a non-constant first integral lies in the fact that the trajectories of the vector field X leave invariant the level sets of the function H, and hence this is a strong constraint on the dynamical behavior of the vector field (e.g. it prevents from the existence of ergodic trajectories on M). Unfortunately, it is generally very difficult to compute a first integral (whose existence is, in fact, a rather non-generic phenomenon). This is the reason why it is necessary to introduce some auxiliary objects to study the integrability properties of a vector field.