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Structural Stability of Planar Homogeneous Polynomial Vector Fields: Applications to Critical Points and to Infinity

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Let H_m be the space of planar homogeneous polynomial vector fields of degree m endowed with the coefficient topology. We characterize the set Ω_m of the vector fields of H_m that are structurally stable with respect to perturbations in H_m and we determine the exact number of the topological equivalence classes in Ω_m. The study of structurally stable homogeneous polynomial vector fields is very useful for understanding some interesting features of inhomogeneous vector fields. Thus, by using this characterization we can do first an extension of the Hartman–Grobman Theorem which allows us to study the critical points of planar analytical vector fields whose k-jets are zero for all $k < m$ under generic assumptions and second the study of the flows of the planar polynomial vector fields in a neighborhood of the infinity also under generic assumptions. © 1996 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

We denote by H_m the set of planar homogeneous polynomial vector fields of degree m; this is, $X \in H_m$ if

$$X = (P, Q) \colon \mathbf{R}^2 \to \mathbf{R}^2,$$

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where P and Q are homogeneous polynomial in the variables x and y of degree m. The system of differential equations associated to X is:

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y). \tag{1}$$

As far as we know the study of homogeneous polynomial vector fields started in 1960 with a paper of Markus [M], where he classified the quadratic homogeneous polynomial vector fields X such that P and Q have no common factor.

Later in 1968 Argemí [A] completed the classification of Markus. Moreover, he furnished the classification of the cubic vector fields that have no common factor. At the same time, he obtained upper and lower bounds for the number of phase portraits of the planar homogeneous polynomial vector fields of degree m which have no common factor.

Subsequent results, relative to an algebraic classification of H_2, can be found in the paper of Date [D] in 1979. There, the author also gives the classification of quadratic vector fields with common factors. This algebraic classification has also been made in a different way by Sibirsky [Si] using algebraic invariants.

In 1990 Cima and Llibre [CL] obtained a topological classification of the cubic homogeneous polynomial vector fields with or without common factors and they present an algorithm for studying the phase portraits of homogeneous polynomial vector fields of degree $m \geqslant 3$. In that paper, one can also find an algebraic classification of H_3 which was extended recently by Collins [Co] to H_m for all $m \geqslant 1$.

One of the aims of this paper is the study of the structurally stable vector fields $X \in H_m$ with respect to perturbations in the space H_m, and to apply it to the study the local phase portrait of the degenerate critical points of planar analytic vector fields, and to study the infinity of the planar polynomial vector fields. Many authors have studied the structural stability for different classes of vector fields on 2-dimensional manifolds.

The first definition of stuctural stability for planar vector fields goes back to Andronov and Pontrjagin [AP], who in 1937 studied the structural stability for analytic vector fields on the closed 2-dimensional disc. Roughly speaking, we say that a vector field X is structurally stable if its phase portrait is topologically equivalent (via a homeomorphism near the identity map called the equivalence homeomorphism) to the phase portrait of all of its neighbors in a suitable topology.

In 1962 Peixoto [P] extended these results by characterizing the C^1-vector fields defined on a compact differentiable 2-manifold without a boundary. Also he showed under his assumptions that the requeriment for the equivalence homeomorphism to lie in a pre-assigned neighborhood of identity is redundant.

In 1982 Kotus *et al.* [KKN] gave sufficient conditions for a C^1-vector field in an open differentiable 2-manifold, N^2, to be structurally stable. Furthermore, they proved that these conditions are necessary if $N^2 = \mathbf{R}^2$. They provided an example which shows that when the 2-manifold is open the requirement for the equivalence homeomorphism to lie near the identity is not redundant.

In 1987 Shafer [S1] considered the set of polynomial vector fields of degree $\leqslant n$ on $\mathbf{R}^2$ and gave sufficient conditions for structural stability when only polynomial perturbations are allowed. He proved that these conditions are necessary with one exception related with the hiperbolicity of limit cycles.

Also Shafer in 1990 [S2] characterized the planar gradient polynomial vector fields which are structurally stable with respect to perturbations in the set of all C^r planar vector fields and in the set of all planar polynomial vector fields. Also, he presented sufficient conditions for structural stability in the set of all planar gradient polynomial vector fields.

In 1993, Jarque and Llibre [JL1] characterized the structurally stable planar Hamiltonian polynomial vector fields with respect to perturbations, first in the set of all C^r planar vector fields, second in the set of all planar polynomial vector fields, and third in the set of all planar Hamiltonian polynomial vector fields. The same authors in [JL2] studied the structural stability of C^r planar Hamiltonian polynomial vector fields in the set of all C^r planar vector fields extending these results to the integrable vector fields.

Recently Artés *et al.* [AKL] complete the classification of the structurally stable planar quadratic polynomial vector fields without limit cycles with respect to perturbations, first in the set of all C^r planar vector fields, second in the set of all planar quadratic vector fields, and third in the set of all compactified planar quadratic vector fields.

We begin by changing system (1) to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. So, the expression of system (1) goes over to

$$\frac{dr}{dt} = r^m f(\theta), \qquad \frac{d\theta}{dt} = r^{m-1} g(\theta),$$

where

$$\begin{aligned} f(\theta) &= \cos\theta P(\cos\theta, \sin\theta) + \sin\theta Q(\cos\theta, \sin\theta), \\ g(\theta) &= \cos\theta Q(\cos\theta, \sin\theta) - \sin\theta P(\cos\theta, \sin\theta). \end{aligned} \tag{2}$$

If we introduce a new time s via $ds/dt = r^{m-1}$, then the above system becomes

$$r' = rf(\theta), \qquad \theta' = g(\theta),$$

where prime denotes derivative with respect to s.

Finally, the change of variable $\rho = r/(1+r)$ transforms (1) into the topological equivalent system given by

$$\rho' = \rho(1-\rho)\, f(\theta), \qquad \theta' = g(\theta), \tag{3}$$

when (ρ, θ) are taken in the open disc $D = \{(\rho, \theta) : 0 \leqslant \rho < 1\}$. Notice that system (3) is also defined for $\rho \geqslant 1$.

Since $\rho' = 0$ when $\rho = 1$, then the boundary of D, $\partial D = \{(\rho, \theta) : \rho = 1\}$ is an invariant circle under the flow of (3). This circle corresponds to the infinity of system (1), and therefore the vector field $E(X)$, associated to the system (3) and defined in an open neighborhood U of $\bar{D}$, is an analytical extension of the vector field X to the infinity. As usual here $\bar{D}$ denotes the closure of D in $\mathbf{R}^2$. Although we are only concerned with the phase portraits of $E(X)$ on the closed disc $\bar{D}$, it will be useful to consider $E(X)$ defined on the neighborhood U. In this way we will be able to apply the standard results about critical points in order to study the local phase portraits of the critical points of $E(X)$ on ∂D.

We shall say that X, $Y \in H_m$ are *topologically equivalent* if there exists a homeomorphism $h: \bar{D} \to \bar{D}$ such that orbits of the flow induced by $E(X)$ are carried onto orbits of the flow induced by $E(Y)$, preserving sense but not necessarily parametrization.

We note that every $X \in H_m$ is specified in some unique way by the $2m+2$ coefficients of P and Q, and hence it may be identified with a unique point in $\mathbf{R}^{2m+2}$. Let us take in H_m the topology induced by the Euclidean norm of $\mathbf{R}^{2m+2}$. Then we say that $X \in H_m$ is *structurally stable* with respect to perturbations in H_m if there exists a neighborhood U of X in H_m such that for all $Y \in U$ we have that X and Y are topologically equivalent.

It is interesting to remark that in the above definition of structural stability we do not say that the equivalence homeomorphism is near the identity map on $\bar{D}$, as is usual in the literature. In Section 3 we shall prove that this condition is redundant in our context.

The following result characterizes the vector fields in H_m that are structurally stable.

Theorem A. *The vector field $X = (P, Q) \in H_m$ is structurally stable with respect to perturbations in H_m if and only if it satisfies one of the following conditions*:

(a) *If $E(X)$ has no critical points on ∂D, then*

$$I_X = \int_0^{2\pi} \frac{f(\theta)}{g(\theta)}\, d\theta \neq 0,$$

where f and g are the functions defined in (2).

(b) *If $E(X)$ has critical points on ∂D, then all these points are hyperbolic.*

We shall prove Theorem A in Section 3.

Let Ω_m be the set of vector fields $X \in H_m$ which are structurally stable with respect to perturbations in H_m and let us denote by C_m the number of topological equivalence classes in Ω_m. To compute the value of C_m it will be necessary to introduce the numbers P_{2t} and I_{2t} defined, according to the parity of m, in the following way. If m is odd, then

$$P_{2t} = \frac{1}{t}\Bigl(2^{2t} - \sum_{\substack{r \mid t \\ r \neq t}} r P_{2r}\Bigr), \qquad I_{2t} = 2^{t+1} - \sum_{\substack{r \mid t \\ r \neq t}} I_{2r}.$$

If m is even, then P_{2t} and I_{2t} are only defined when t is odd by:

$$P_{2t} = \frac{1}{t}\Bigl(2^{t} - \sum_{\substack{r \mid t \\ r \neq t}} r P_{2r}\Bigr), \qquad I_{2t} = 2^{(t+1)/2} - \sum_{\substack{r \mid t \\ r \neq t}} I_{2r}.$$

The following result determines exactly the value of C_m.

THEOREM B. *The value of C_m is given by*

$$C_m = \begin{cases} 1 + \frac{1}{2} \displaystyle\sum_{l=1}^{(m+1)/2} \sum_{t \mid l} (P_{2t} + I_{2t}) & \text{if } m \text{ is odd,} \\ -1 + \frac{1}{2} \displaystyle\sum_{l=0}^{m/2} \sum_{t \mid 2l+1} (P_{2t} + I_{2t}) & \text{if } m \text{ is even.} \end{cases}$$

We shall prove Theorem B in Section 4. See the growth of C_m with respect to m in Table I.

We say that two analytical vector fields X, Y are *locally topologically equivalent at the origin* if there exist two neighborhoods U and V of the origin and a homeomorphism $h: U \to V$ that carries orbits of the flow induced by X onto orbits of the flow induced by Y, preserving sense but not necessarily parametrization.

TABLE I

Some Values of C_m for $m \leqslant 100$

m	C_m	m	C_m
1	5	2	5
3	14	4	13
5	34	6	31
7	85	8	77
9	221	10	203
11	635	12	583
13	1935	14	1807
15	6306	16	5919
17	21390	18	20229
19	74898	20	71193
29	48988442	30	47366899
39	3.732086 $E+10$	40	3.639319 $E+10$
49	3.045374 $E+13$	50	2.984774 $E+13$
59	2.592225 $E+16$	60	2.549214 $E+16$
69	2.271252 $E+19$	70	2.238934 $E+19$
79	2.032408 $E+22$	80	2.007093 $E+22$
89	1.848100 $E+25$	90	1.827632 $E+25$
99	1.701862 $E+28$	100	1.684894 $E+28$

Let $X = \sum_{i \geqslant m} X_i$ where each X_i is an homogeneous polynomial vector field of degree i and $m \geqslant 1$. The next theorem allows us to know the phase portrait of the analytical vector field X in a neighborhood of the origin, when X_m is structurally stable with respect to perturbations in H_m.

THEOREM C. *Let $X = (\sum_{i \geqslant m} P_i, \sum_{i \geqslant m} Q_i)$ be an analytical vector field where P_i and Q_i are homogeneous polynomials of degree i and let $X_m = (P_m, Q_m)$. If X_m is structurally stable with respect to perturbations in H_m, then the phase portraits of X and X_m are locally topologically equivalent at the origin.*

In fact from the proof of Theorem C in Section 5 it is easy to see that Theorem C can be extended to C^{m+1} vector fields in a neighborhood of the origin. A different result but in the same direction of Theorem C was given by Coleman [C].

In Section 5, we also prove that the converse of Theorem C is not true for any $m \geqslant 1$. Consequently there is a number greater than C_m of topological equivalence classes in H_m whose phase portraits remain locally topologically equivalent at the origin by perturbations $X = (\sum_{i > m} P_i, \sum_{i > m} Q_i)$, where P_i and Q_i are homogeneous polynomial of degree i.

From Theorem C we also obtain that C_m is the exact number of the topological equivalence classes in H_m whose phase portraits remain locally topologically equivalent at the origin if we consider perturbations

$X=(\sum_{i\geqslant m} P_i, \sum_{i\geqslant m} Q_i)$, where P_i and Q_i are homogeneous polynomials of degree i and $X_m=(P_m, Q_m)$ is sufficiently close to 0 in H_m.

Let $m\in\mathbf{N}$ and let A_m be the set of all analytical vector fields $X=\sum_{k\geqslant 0} X_k$ such that their k-jets X_k are zero for $k<m$. We endow A_m with the finest topology such that the inclusion $i\colon H_m\to A_m$ is a continous function. Then we say that $X\in A_m$ is *locally structurally stable at the origin* if there exists a neighborhood W of X in A_m such that for each $Y\in W$, X, and Y are locally topologically equivalent at the origin.

COROLLARY D. *The number of classes of local topological equivalence at the origin in the set A_m that are locally structurally stable at the origin is C_m.*

For $m=1$ we note that the local phase portraits of Corollary D are just the phase portraits of Hartman–Grobman Theorem in the plane. In fact, Theorem C and Corollary D extend the Hartman–Grobman Theorem to arbitrary $m\geqslant 1$.

Next we shall present a study of the flow near the infinity for the planar polynomial vector fields of degree m, in a similar way as we studied the flow near a critical point in Theorem C and Corollary D.

We say that two polynomial vector fields X and Y are *locally topologically equivalent at infinity* if there exist two neighborhoods U and V of the infinity ∂D and a homeomorphism $h\colon U\to V$ that carries orbits of the flow induced by X onto orbits of the flow induced by Y, preserving sense but not necessarily parametrization.

THEOREM E. *Let $X=(\sum_{i=0}^{m} P_i, \sum_{i=0}^{m} Q_i)$ a polynomial vector field where P_i and Q_i are homogeneous polynomials of degree i and let $X_m=(P_m, Q_m)$. If X_m is structurally stable with respect to perturbations in H_m, then the phase portraits of X and X_m are locally topologically equivalent at infinity.*

The proof of Theorem E is in Section 6 and its converse is not true. A result close to Theorem E was given by Cima and Llibre in [CL].

Let B_m be the set of all polynomial vector fields $X=X_0+X_1+\cdots+X_m$ such that their m-jet X_m is not zero. We endow B_m with the coefficient topology. Then, we say that $X\in B_m$ is *locally structurally stable at infinity* if there exists a neighborhood W of X in B_m such that for each $Y\in W$, X and Y are locally topologically equivalent at infinity.

COROLLARY F. *The number of classes of local topological equivalence at infinity in the set B_m that are locally structurally stable at infinity is C_m.*

This work is organized as follows. In Section 2 we present the main results of Argemí [A] (see also [CL]) about an alghoritm that allows to determine the phase portraits of the vector fields in H_m. In Section 3 we prove Theorem A that characterizes the vector fields $X \in H_m$ that are structurally stable with respect to perturbations in H_m. In Section 4 the largest of this paper, we compute the number C_m of topological equivalence classes of the structurally stable vector fields in H_m with respect to perturbations in H_m. In Section 5 we study the phase portraits of analytical vector fields in a neighborhood of the origin. Finally, in Section 6 we study the phase portraits of polynomial vector fields in a neighborhood of the infinity.

2. Phase-Portraits

Let (x_0, y_0) be a critical point of a vector field in the plane. We say that (x_0, y_0) is *elemental* if there exists at least one nonzero eigenvalue of its linear part. The critical point (x_0, y_0) is called *hyperbolic* if the eigenvalues of its linear part have nonzero real parts.

The phase portraits of the homogeneous polynomial vector fields had been studied in [A] and [CL]. We present here their main results.

Proposition 1. *Let $X \in H_m$. Assume that $E(X)$ has no critical points in ∂D and let I_X be defined as in Theorem* A. *Then the phase portrait of $E(X)$ in D is*:

(a) *a global center if and only if $I_X = 0$.*

(b) *a global stable (respectively unstable) focus if and only if $I_X \cdot \theta' < 0$ (respectively $I_X \cdot \theta' > 0$).*

Proof. See Proposition 4.2 of [CL]. ∎

Proposition 2. *Let $X \in H_m$ and suppose that* $(0, 0)$ *is an isolated critical point of X. Assume that $E(X)$ has critical points in ∂D. Then the following holds*:

(a) *If θ^* is a zero of $g(\theta)$ (where g is defined in* (2)), *then the straight line of slope* $\tan \theta^*$ *which passes through the origin is invariant under the flow induced by $E(X)$.*

(b) *$E(X)$ has no limit cycles in D.*

(c) *The critical points of $E(X)$ on ∂D are all elemental and they are nodes, saddles, or saddle–nodes. A critical point* $(1, \theta)$ *on ∂D is a saddle–node if and only if θ is a zero of $g(\theta)$ of even multiplicity. Furthermore, the separatrix associated to eigenvalue* 0 *is contained in ∂D (see Fig.* 4.2 *of* [CL]).

Proof. See Proposition 4.1 of [CL]. ∎

Let $X \in H_m$ be a vector field under the assumptions of Proposition 2. Then, the phase portrait of $E(X)$ in $\bar{D}$ can be obtained through the union of an even number ($\leqslant 2m+2$) of elliptic, hyperbolic and parabolic sectors (see Fig. 4.3 of [CL]). The boundaries of these sectors which are not contained in ∂D, correspond to straight lines of slope $\tan \theta^*$, where θ^* is a zero of $g(\theta)$.

We note that so far we have identified the origin of (1) with only one point in D, as it is usual in polar coordinates. Nevertheless $\rho = 0$ is an invariant circle under the flow induced by (3), and the number of critical points of $E(X)$ on $\rho = 0$ is also determined by the zeros of $g(\theta)$.

The following proposition shows the similarity between the flow induced by (3) in a neighborhood of $\rho = 0$ (origin of (1)) and in a neighborhood of $\rho = 1$ (infinity of (1)).

Proposition 3. *Let $X \in H_m$ and suppose that* $(0, 0)$ *is an isolated critical point of X.*

(a) *If $E(X)$ has no critical points in $\rho = 1$, then $\rho = 0$ is an isolated periodic orbit for the flow induced by* (3) *if and only if $I_X \neq 0$.*

(b) *If $E(X)$ has critical points on $\rho = 1$, then $(1, \theta)$ is a hyperbolic critical point if and only if the critical point $(0, \theta)$ is also hyperbolic.*

Proof. By the change of variable $\sigma = 1 - \rho$, system (3) becomes

$$\sigma' = -\sigma(1-\sigma) f(\theta), \qquad \theta' = g(\theta), \tag{4}$$

and $\rho = 0$ goes over $\sigma = 1$.

If $E(X)$ has no critical points in $\rho = 1$, then system (4) has no critical points in $\sigma = 1$, since in both cases these critical points are determined by the zeros of $g(\theta) = 0$. Now if we apply the same arguments of Proposition 1 to system (4), then $\sigma = 1$ is a isolated periodic orbit if and only if

$$J = -\int_0^{2\pi} \frac{f(\theta)}{g(\theta)} \, d\theta \neq 0.$$

As $J = -I_X$ statement (a) of the proposition follows.

In order to prove (b) we observe that the Jacobian matrix of $E(X)$ in a critical point $(1, \theta^*)$ is

$$\begin{pmatrix} -f(\theta^*) & 0 \\ 0 & g'(\theta^*) \end{pmatrix}. \tag{5}$$

On the other hand, the critical point $(0, \theta^*)$ of $E(X)$ corresponds to the critical point $(1, \theta^*)$ of (4), whose Jacobian matrix is

$$\begin{pmatrix} f(\theta^*) & 0 \\ 0 & g'(\theta^*) \end{pmatrix}. \tag{6}$$

From (5) and (6) we have that $(1, \theta^*)$ is hyperbolic for (3) if and only if it is hyperbolic for (4) and, therefore, the proposition follows. ∎

We note that although the critical points $(0, \theta^*)$ and $(1, \theta^*)$ of $E(X)$ are simultaneously both hyperbolic and nonhyperbolic, these points are topologically different. For instance, from (5) and (6) we obtain that if $(1, \theta^*)$ is a saddle, then $(0, \theta^*)$ is a node.

3. Structural Stability

In this section we characterize the vector fields $X \in H_m$ which are structurally stable with respect to perturbations in H_m. First we consider the sphere $\mathbf{S}^2$ obtained by the union of two copies of the disc D, when we identify the points of ∂D. Then we can define on $\mathbf{S}^2$ an analytical extension of $X \in H_m$, and we denote the analytic vector field on $\mathbf{S}^2$ by $\xi(X)$. The flow induced by $\xi(X)$ has two copies of the flow induced by X, one on the northern hemisphere and the other on the southern hemisphere of $\mathbf{S}^2$. This extension is in fact the Poincaré compactification of X (see [G] or [So]).

Through the study of the phase portraits of the vector fields in H_m (see Section 2), we obtain that the extension $\xi(X)$ in a neighborhood of the equator $\mathbf{S}^1 \subset \mathbf{S}^2$ is topologically equivalent to the extension $E(X)$ in a neighborhood of ∂D. We note that, from Proposition 2, the local phase portrait in a critical point of $E(X)$ in ∂D is the same as the local phase portrait in the corresponding critical point of $\xi(X)$ in $\mathbf{S}^1$. Therefore the definition of structural stability of Section 1 is equivalent to the next definition. Let $X \in H_m$. We say that X is structurally stable if there exists a neighborhood V of X in H_m such that for all $Y \in V$ we have that $\xi(X)$ and $\xi(Y)$ are topologically equivalent; that is, there exists a homeomorphism of $\mathbf{S}^2$ having invariant the equator $\mathbf{S}^1$ that carries orbits of the flow induced by $\xi(X)$ onto orbits of the flow induced by $\xi(Y)$, preserving sense but not necessarily parametrization.

Peixoto in [P] showed that on an orientable differentiable compact connected 2-manifold without boundary, if a C^1 vector field X is equivalent to all vector fields in a neighborhood U of X in the C^1 topology, then the equivalence homeomorphism between X and any vector field in U can be chosen sufficiently close to the identity map.

Since the coefficient topology in H_m is equivalent to the C^1 topology (for more details, see [DS]) and $\mathbf{S}^2$ is a manifold under the assumptions of the result of Peixoto, we have that the requirement that the equivalence homeomorphism be near to the identity map is redundant in our second definition of the structural stability. From the equivalence between the two definitions, we deduce that the above condition is also redundant in the first definition of structural stability.

Now, we prove the result that characterizes the vector fields which are structurally stable with respect to perturbations in H_m.

Proof of Theorem A. First, we assume that X is structurally stable with respect to perturbations in H_m and we prove that one of the statements (a) or (b) holds.

From Proposition 1, if $E(X)$ has no critical points in ∂D, then the flow induced by $E(X)$ in $\bar{D}$ is determined by the sign of $I_X \cdot \theta'$. When $I_X = 0$, the phase portrait of X in D is a global center and there exists a vector field Y in any neighborhood of X in H_m such that the phase portrait of Y in D is a global focus. Therefore X is not structurally stable with respect to perturbations in H_m and the statement (a) holds.

Now we suppose that $E(X)$ has critical points in ∂D. Since X is structurally stable with respect to perturbations in H_m, it has no straight lines of critical points and from (3) it follows that the origin is an isolated critical point of X. Then we can apply Proposition 2 and obtain that the phase portrait of X is completely determined by the behavior of the flow induced by X in a neighborhood of ∂D. Furthermore the critical points of $E(X)$ in ∂D are determined by the zeros of $g(\theta)$.

If $g(\theta)$ has multiple zeros, then we can choose in any neighborhood of X in H_m a vector field Y such that $E(Y)$ does not have the same number of critical points in ∂D than $E(Y)$. So $E(X)$ and $E(Y)$ are not topologically equivalent and X is not structurally stable with respect to perturbations in H_m. Then, from the expression of Jacobian matrix of $E(X)$ in a critical point $(\rho, \theta) = (1, \theta^*)$ (see (5)), in order to prove statement (b) it is sufficient to prove that $f(\theta^*) \neq 0$, for each zero θ^* of $g(\theta)$. But if $f(\theta^*) = g(\theta^*) = 0$, then the straight line of slope $\tan \theta^*$ is formed by critical points of X. Therefore X is not structurally stable with respect to perturbations in H_m and (b) follows.

Next we prove that if $E(X)$ satisfies (a) or (b) then X is structurally stable with respect to perturbations in H_m. First we assume that $E(X)$ has no critical points in ∂D (that is, $g(\theta)$ has a constant sign) and that $I_X \neq 0$. Then there exists a neighborhood U of X in H_m, such that if $Y \in U$ we have that $E(Y)$ has no critical points in ∂D and furthermore $\operatorname{sign}(I_X) = \operatorname{sign}(I_Y)$ and $\operatorname{sign}(g_X) = \operatorname{sign}(g_Y)$, where g_X and g_Y are the functions defined in (2) through the vector fields X and Y respectively. Since these signs determine the global phase portrait of $E(X)$ and $E(Y)$ (see Proposition 1), we conclude that these vector fields are topologically equivalent. Hence X is structurally stable with respect to perturbations in H_m.

Now we assume that $E(X)$ has critical points on ∂D, $(1, \theta_i)$, for $i=1, ..., s$, and all these points are hyperbolic. If we consider the Jacobian matrix of $E(X)$ in these points (see (5)), then we obtain that $g'(\theta_i) \neq 0$ and $f(\theta_i) \neq 0$, for $i=1, ..., s$. Therefore the zeros of g are simple and there exists a neighborhood U of X in H_m such that if $Y \in U$ we get that $E(Y)$ has exactly s critical points on ∂D, $(1, \overline{\theta_i})$, for $i=1, ..., s$. Furthermore we can choose the above neighborhood in such a way that $\text{sign}(g'_Y(\overline{\theta_i})) = \text{sign}(g'_X(\theta_i))$ and $\text{sign}(f_Y(\overline{\theta_i})) = \text{sign}(f_X(\theta_i))$, for each $i=1, ..., s$, where g_X and f_X (respectively, g_Y and f_Y) are the functions defined in (2) by using the components of X (respectively Y). Thus the local phase portraits of $E(X)$ and $E(Y)$ at $(1, \theta_i)$ and $(1, \overline{\theta_i})$, respectively, are topologically equivalent. From Proposition 2, the behavior of the critical points on ∂D determines the global phase portrait of $E(X)$ and $E(Y)$ on $\bar{D}$. Hence $E(X)$ and $E(Y)$ are topologically equivalent, that is, X is structurally stable with respect to perturbations in H_m and the proposition follows. ∎

4. The Calculation of C_m

Let Ω_m be the set of all structurally stable vector fields with respect to perturbations in H_m. In this section we prove Theorem B which determines the exact number of topological equivalence classes in Ω_m.

From Theorem A, if $X \in \Omega_m$, we know that the function $g(\theta)$ has at most a finite number of zeros in $[-\pi/2, 3\pi/2)$ and that these zeros are simple. Then, doing a suitable rotation if it is necessary, we can assume that $\cos \theta^* \neq 0$ for each zero θ^* of $g(\theta)$. This assumption is equivalent to suppose that the coefficient of the monomial y^m in the polynomial $P(x, y)$ is not zero. Under this assumption, the functions f and g of (2) can be expressed by

$$\begin{aligned} f(\theta) &= [P(1, \lambda) + \lambda Q(1, \lambda)] \cos^{m+1} \theta = \bar{f}(\lambda) \cos^{m+1} \theta, \\ g(\theta) &= [Q(1, \lambda) - \lambda P(1, \lambda)] \cos^{m+1} \theta = \bar{g}(\lambda) \cos^{m+1} \theta, \end{aligned} \tag{7}$$

where $\lambda = \tan \theta$.

The function $\bar{g}(\lambda)$ is a polynomial of degree $m+1$ and its zeros determine those of $g(\theta)$. Each zero λ^* of $\bar{g}(\lambda)$ defines an invariant straight line through the origin of (3) with slope λ^*, and two zeros θ^* and $\theta^* + \pi$ of $g(\theta)$ in $(-\pi/2, 3\pi/2)$ such that $\tan \theta^* = \lambda^*$.

PROPOSITION 4. *Let $X=(P, Q)\in\Omega_m$ and assume that the coefficient of the monomial y^m in the polynomial $P(x, y)$ is not zero. Then there exists an even number $n=2k$ of zeros of $g(\theta)$ in $(-\pi/2, 3\pi/2)$ with $k\leqslant m+1$ and $k\equiv m+1 \pmod 2$.*

Proof. From (7) and the above comment, it follows that if k is the number of zeros of $\bar{g}(\lambda)$ then $2k$ is the number of zeros of $g(\theta)$ in $(-\pi/2, 3\pi/2)$. Because of $X\in\Omega_m$ these zeros are simple and since $\bar{g}$ is of degree $m+1$ the proposition follows. ∎

Henceforth, let

$$\Omega_m^n=\{X\in\Omega_m : E(X) \text{ has } n \text{ critical points in } \partial D\}.$$

COROLLARY 5. *Let $m\in\mathbf{N}$.*

(a) *If m is odd then $\Omega_m=\bigcup_{k\in J_m}\Omega_m^{2k}$, where*

$$J_m=\left\{k=2l : 0\leqslant l\leqslant\frac{m+1}{2}\right\}.$$

(b) *If m is even then $\Omega_m=\bigcup_{k\in J_m}\Omega_m^{2k}$, where*

$$J_m=\left\{k=2l+1 : 0\leqslant l\leqslant\frac{m}{2}\right\}.$$

Proof. It is an easy consequence of Proposition 4. ∎

From the study of the phase portraits of the vector fields in H_m (see Section 2), we know that two topologically equivalent vector fields $X, Y\in\Omega_m$ have the same number of critical points in ∂D. Then if we denote by C_m^k the number of topological equivalence classes in Ω_m^{2k}, from Corollary 5 it follows that $C_m=\sum_{k\in J_m}C_m^k$. Therefore our next goal is to compute the value of C_m^k for every $k\in J_m$.

If $k=0$ it is immediate from Proposition 1 and Theorem A that $C_m^0=2$ (a global stable focus and a global unstable focus). So in the rest of this section we shall discuss the case $k\neq 0$.

PROPOSITION 6. *Let $X=(P; Q)\in\Omega_m^{2k}$ with $m\in\mathbf{N}$ and $0\neq k\in J_m$. Let $g(\theta)$ be the function associated to X in (2). Denote by θ_i, $i=1, ..., 2k$, the ordered counterclockwise zeros of $g(\theta)$ in $(-\pi/2, 3\pi/2)$. Then it follows that*

(a) *$f(\theta_i)\neq 0$ and $g'(\theta_i)\neq 0$, for $i=1, ..., 2k$.*

(b) *$g'(\theta_i)\, g'(\theta_{i+1})<0$, for $i=1, ..., 2k-1$.*

Proof. According to (3) the critical points of $E(X)$ in ∂D are $(1, \theta_i)$, $i = 1, ..., 2k$. Now, since X is structurally stable with respect to perturbations in H_m, it follows from Theorem A that these critical points are hyperbolic. Lastly from the expression of the Jacobian matrix of $E(X)$ at the critical point $(1, \theta_i)$ given in (5) we get (a).

On the other hand, if $\bar{\theta} < \theta^*$ are zeros of $g(\theta)$ such that $g'(\bar{\theta})\, g'(\theta^*) > 0$ then there exists a zero of $g(\theta)$ in $(\bar{\theta}, \theta^*)$. Hence, since θ_i and θ_{i+1} are consecutive zeros of $g(\theta)$, the statement (b) follows using (a). ∎

Now let S be the set of all sequences $(\sigma, v) = \{(\sigma_i, v_i)\}_{i \in \mathbf{Z}}$ such that $\sigma_i, v_i \in \{-1, 1\}$ and $\sigma_i \sigma_{i+1} < 0$ for all $i \in \mathbf{Z}$. For each $k \in \mathbf{N}$, we denote

$$S^{2k} = \{(\sigma, v) \in S : (\sigma_i, v_i) = (\sigma_{i+2k}, v_{i+2k}), \text{ for all } i \in \mathbf{Z}\}.$$

A sequence (σ, v) is *periodic of period p* (or *p-periodic*) if p is the smallest natural number such that $(\sigma_i, v_i) = (\sigma_{i+p}, v_{i+p})$ for all $i \in \mathbf{Z}$. So all the sequences in S^{2k} are periodic and their period p is an even divisor of $2k$. Obviously each sequence $(\sigma, v) \in S^{2k}$ is completely determined if the elements (σ_i, v_i) are given for $i = 1, ..., p$.

From Proposition 6 we can associate a sequence of S^{2k} to each vector field $X \in \Omega_m^{2k}$ by taking

$$(\sigma_i, v_i) = (\operatorname{sign}(g'(\theta_i)), \operatorname{sign}(f(\theta_i))), \qquad i = 1, ..., 2k, \tag{8}$$

where θ_i, for $i = 1, ..., 2k$ are the ordered counterclockwise zeros of $g(\theta)$ in $(-\pi/2, 3\pi/2)$ and $\operatorname{sign}(x) = x/|x|$ if $x \neq 0$, $\operatorname{sign}(0) = 0$. We say that $(\sigma, v) \in S^{2k}$ is *m-admissible* if there exists $X \in \Omega_m^{2k}$ satisfying (8). Denote by S_m^{2k} the set of all sequences in S^{2k} that are m-admissible. From Corollary 5 it follows that $S_m^{2k} \neq \varnothing$ if and only if $k \in J_m$.

PROPOSITION 7. *Let* $X = (P, Q) \in \Omega_m^{2k}$ *with* $m \in \mathbf{N}$ *and* $0 \neq k \in J_m$. *Let* $(\sigma, v) \in S_m^{2k}$ *the sequence associated to X according to* (8).

(a) *If* $\bar{f}$ *and* $\bar{g}$ *are the functions defined in* (7) *and* $\lambda_1 < \cdots < \lambda_k$ *are the ordered real zeros of* $\bar{g}$, *then* $(\sigma_i, v_i) = (\operatorname{sign}(\bar{g}'(\lambda_i)), \operatorname{sign}(\bar{f}(\lambda_i)))$ *for* $i = 1, ..., k$.

(b) $(\sigma_{i+k}, v_{i+k}) = (-1)^{m+1} (\sigma_i, v_i)$ *for* $i = 1, ..., k$.

Proof. If θ_i for $i = 1, ..., 2k$ are the ordered counterclockwise zeros of $g(\theta)$ in $(-\pi/2, 3\pi/2)$ and λ_i for $i = 1, ..., k$ are the ordered increasing real zeros of $\bar{g}(\lambda)$, then according to (7):

$$\begin{aligned} \operatorname{sign}(f(\theta_i)) &= \operatorname{sign}(\bar{f}(\lambda_i)) \operatorname{sign}(\cos^{m+1} \theta_i), \\ \operatorname{sign}(f(\theta_{i+k})) &= \operatorname{sign}(\bar{f}(\lambda_i)) \operatorname{sign}(\cos^{m+1} \theta_{i+k}), \end{aligned} \tag{9}$$

since θ_i and $\theta_{i+k} = \theta_i + \pi$ correspond to the same zero λ_i of $\bar{g}$.

On the other hand, again from (7) it follows that

$$g'(\theta_i) = \bar{g}'(\lambda_i)(1+\lambda_i^2)\cos^{m+1}\theta_i,$$
$$g'(\theta_{i+k}) = \bar{g}'(\lambda_i)(1+\lambda_i^2)\cos^{m+1}\theta_{i+k},$$

for $i = 1, ..., k$. So we obtain for g' a relation similar to (9)

$$\begin{aligned} \operatorname{sign}(g'(\theta_i)) &= \operatorname{sign}(\bar{g}'(\lambda_i))\operatorname{sign}(\cos^{m+1}\theta_i), \\ \operatorname{sign}(g'(\theta_{i+k})) &= \operatorname{sign}(\bar{g}'(\lambda_i))\operatorname{sign}(\cos^{m+1}\theta_{i+k}), \end{aligned} \tag{10}$$

for $i = 1, ..., k$. From (8), (9), and (10) the proposition follows easily. ∎

Since each m-admissible sequence satisfies Proposition 7(b), it follows that if m is odd then all the sequences in S_m^{2k} have a period less or equal than k. In any case, a sequence $(\sigma, v) \in S_m^{2k}$ is completely determined if the elements (σ_i, v_i) are given for $i = 1, ..., k$.

The next lemmas allow us to set Proposition 10 which characterizes the m-admissible sequences in S^{2k}. In what follows, we shall call the coefficient of the monomial of highest degree of a polynomial the *principal coefficient* of the polynomial.

LEMMA 8. *Let $m \in \mathbf{N}$ and $0 \neq k \in J_m$. Assume that $(\sigma, v) \in S^{2k}$ and verifies $(\sigma_{i+k}, v_{i+k}) = (-1)^{m+1}(\sigma_i, v_i)$ for $i = 1, ..., k$. Then (σ, v) is m-admissible if and only if there exist a polynomial $R(\lambda)$ of degree $m+1$ which has just k real zeros $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and a polynomial $S(\lambda)$ of degree m such that their principal coefficients have opposite sign and they satisfy*

$$\sigma_i = \operatorname{sign}(R'(\lambda_i)), \qquad v_i = \operatorname{sign}(S(\lambda_i)), \tag{11}$$

for $i = 1, ..., k$.

Proof. If (σ, v) is m-admissible then, there exists $X \in \Omega_m^{2k}$ satisfying (8). From Proposition 7(a) it follows that

$$\sigma_i = \operatorname{sign}(\bar{g}'(\lambda_i)), \qquad v_i = \operatorname{sign}(\bar{f}(\lambda_i)), \tag{12}$$

where $\bar{f}$ and $\bar{g}$ are the functions defined in (7) and $\lambda_1 < \cdots < \lambda_k$ are the real zeros of $\bar{g}(\lambda)$. Define $R(\lambda) = \bar{g}(\lambda)$ which is a polynomial of degree $m+1$ with k real zeros. To obtain S we note that if λ_i is a zero of $\bar{g}$ then it is evident from (7) that $\bar{f}(\lambda_i) = (1+\lambda_i^2)\, P(1, \lambda_i)$, and so

$$\operatorname{sign}(\bar{f}(\lambda_i)) = \operatorname{sign}(P(1, \lambda_i)) \qquad \text{for} \quad i = 1, ..., k. \tag{13}$$

Then $S(\lambda) = P(1, \lambda)$ is a polynomial of degree m and we deduce from (7) that the principal coefficients of S and R have opposite signs. The condition (11) follows from (12) and (13) and the necessary condition in the lemma is proved.

Now, we suppose that for a sequence $(\sigma, \nu) \in S^{2k}$ there exist polynomials R and S of degree $m+1$ and m respectively, satisfying (11) and such that their principal coefficients have opposite signs. Then, in order to prove the sufficient condition of the lemma, we have to find a vector field $X \in \Omega_m^{2k}$ in such a way that (σ, ν) is the sequence associated to X according to (8).

Let $P(x, y)$ and $Q(x, y)$ be the homogeneous polynomials of degree m. Define $X=(P, Q)$. Then the function $\bar{g}(\lambda)$ associated to X is $R(\lambda)$ and the signs of $\bar{f}(\lambda)$ and $S(\lambda)$ coincide in the zeros of $\bar{g}$. Thus from (11) we obtain

$$(\sigma_i, \nu_i) = (\operatorname{sign}(\bar{g}'(\lambda_i)), \operatorname{sign}(\bar{f}(\lambda_i))) \qquad \text{for} \quad i=1, ..., k.$$

Finally, by taking into account that $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m+1}(\sigma_i, \nu_i)$ for $i=1, ..., k$ and according to Proposition 7, it follows that (σ, ν) is the sequence of S^{2k} associated to X and, hence (σ, ν) is m-admissible. ∎

LEMMA 9. *Let $r \in \mathbf{N}$, $\delta \in \{-1, 1\}$ and a real interval $[c, d]$. Then there exists a polynomial $P(\lambda) = \lambda^r + a_1\lambda^{r-1} + \cdots + a_r$ such that $\operatorname{sign}(P(\lambda)) = \delta$ for every $\lambda \in [c, d]$.*

Proof. Let r be an even number. If $\delta = 1$, take $P(\lambda) = (\lambda^2+1)^{r/2}$, and if $\delta = -1$ define $P(\lambda) = (\lambda^2+1)^{(r-2)/2}(\lambda+1-c)(\lambda-d-1)$.

Let now r be an odd number. If $\delta = 1$, define $P(\lambda) = (\lambda^2+1)^{(r-1)/2} \times (\lambda+1-c)$. In the other case, take $P(\lambda) = (\lambda^2+1)^{(r-1)/2}(\lambda-1-d)$. ∎

The next proposition characterizes the sequences in S^{2k} that are m-admissible. From Proposition 7(b) it is only neccesary to consider those sequences that satisfy the assumptions of Lemma 8.

PROPOSITION 10. *Let $m \in \mathbf{N}$, and $0 \neq k \in J_m$. Assume that $(\sigma, \nu) \in S^{2k}$ and verifies $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m+1}(\sigma_i, \nu_i)$, for $i=1, ..., k$.*

(a) *If $k < m+1$, then (σ, ν) is m-admissible.*

(b) *If $k = m+1$, then (σ, ν) is m-admissible if and only if there exists $j \in \mathbf{Z}$ such that $\sigma_j \neq \nu_j$.*

Proof. First we shall prove (a). From Lemma 8 it is sufficient to find a polynomial R of degree $m+1$ with k real zeros $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and a polynomial S of degree m, satisfying

$$\operatorname{sign}(R'(\lambda_i)) = \sigma_i, \qquad i=1, ..., k, \tag{14}$$

$$\operatorname{sign}(S(\lambda_i)) = \nu_i, \qquad i=1, ..., k, \tag{15}$$

and such that their principal coefficients have opposite signs. To determine these polynomials we take k arbitrary real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and define

$$h(\lambda) = \prod_{j=1}^{k} (\lambda - \lambda_j).$$

Since $k \in J_m$, from Proposition 4 and Corollary 5, it follows that $(m+1-k)/2 \in \mathbf{N}$, and hence $R(\lambda) = a(\lambda^2+1)^{(m+1-k)/2}\, h(\lambda)$ with $a \in \mathbf{R}$, is a polynomial of degree $m+1$ and its real zeros are $\lambda_1, ..., \lambda_k$. Furthermore since

$$R'(\lambda_i) = a(\lambda_i^2+1)^{(m+1-k)/2} \prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_i - \lambda_j), \qquad i = 1, ..., k,$$

if we choose $a = \pm 1$ in such a way that $\operatorname{sign}(ah'(\lambda_1)) = \sigma_1$, then R satisfies (14).

Now to determine S we choose $\mu \in (\lambda_j, \lambda_{j+1})$ for each $j \in \{1, ..., k\}$ such that $v_j \neq v_{j+1}$. So we obtain p real numbers $\mu_1 < \mu_2 < \cdots < \mu_p$ where p is the number of changes of sign in the sequence $\{v_1, ..., v_k\}$. Next we define

$$t(\lambda) = \prod_{i=1}^{p} (\lambda - \mu_i).$$

Let $P(\lambda)$ be a polynomial satisfying Lemma 9 with $r = m - p$, $c = \lambda_1$, $d = \lambda_k$ and $\delta = \pm 1$ such that $\operatorname{sign}(at(\lambda_1)\delta) = -v_1$. Then we define $S(\lambda) = -at(\lambda)\, P(\lambda)$. This is a polynomial of degree m that satisfies (15) and such that its principal coefficient is $-a$. Since the principal coefficient of $R(\lambda)$ is a, the statement (a) of the proposition is proved.

To prove (b) we note that if $p < m$ we can proceed as in (a). Therefore we assume that $p = m$. Since $k = m+1$ each polynomial R verifying (14) has exactly $m+1$ real zeros and hence it has the form

$$R(\lambda) = a \prod_{j=1}^{m+1} (\lambda - \lambda_j), \qquad a \in \mathbf{R}.$$

In addition, since the number p of changes of sign in the sequence $\{v_1, ..., v_{m+1}\}$ is exactly m, if S is a polynomial of degree m satisfying (15) then all its zeros are real and they belong to the interval $[\lambda_1, \lambda_{m+1}]$. Consequently, S is neccesarily of the form

$$S(\lambda) = b \prod_{j=1}^{m} (\lambda - \mu_j), \qquad \lambda_1 < \mu_1 < \lambda_2 \cdots < \lambda_m < \mu_m < \lambda_{m+1}.$$

Finally it follows easily that $\mathrm{sign}(R'(\lambda_1)) = \mathrm{sign}((-1)^m a)$ and $\mathrm{sign}(S(\lambda_1)) = \mathrm{sign}((-1)^m b)$. So, if (14) and (15) are assumed, then $\mathrm{sign}(a) \neq \mathrm{sign}(b)$ if and only if $\sigma_1 \neq v_1$. Hence the proposition holds. ∎

After associating a sequence of S_m^{2k} to each vector field $X \in \Omega_m^{2k}$ we establish the equivalence relation in S_m^{2k} induced by the topological equivalence relation in Ω_m^{2k}.

PROPOSITION 11. *Let $m \in \mathbf{N}$, $0 \neq k \in J_m$ and $X_1, X_2 \in \Omega_m^{2k}$. Denote by (σ, v), $(\sigma', v') \in S_m^{2k}$ the sequences associated to X_1 and X_2 respectively. Then X_1 and X_2 are topologically equivalent if and only if there exists $\tau \in \mathbf{N}$ $(0 \leqslant \tau \leqslant 2k-1)$ such that one of the next conditions holds*:

$$(\sigma_i, v_i) = (\sigma'_{i+\tau}, v'_{i+\tau}) \qquad \text{for all} \quad i \in \mathbf{Z}, \tag{16}$$

$$(\sigma_i, v_i) = (\sigma'_{2k-i+1+\tau}, v'_{2k-i+1+\tau}) \qquad \text{for all} \quad i \in \mathbf{Z}. \tag{17}$$

Proof. First we consider that X_1 and X_2 are topologically equivalent. Hence there exists a homeomorphism $h: \bar{D} \to \bar{D}$ such that it carries orbits of the flow induced by $E(X_1)$ onto orbits of the flow induced by $E(X_2)$, preserving sense but not necessarily parametrization. Then $h(\partial D) = \partial D$ and so $h|_{\partial D}$ is a homeomorphism of this circle.

Denote by $(1, \theta_i)$ and $(1, \theta'_i)$ for $i = 1, ..., 2k$ the ordered counterclockwise critical points in ∂D of $E(X_1)$ and $E(X_2)$ respectively. The equivalence homeomorphism carries critical points onto critical points preserving the local phase portrait. Therefore, if $h(1, \theta_i) = (1, \theta'_r)$, the local phase portrait of $E(X_1)$ in a neighborhood of $(1, \theta_i)$ is equivalent to the local phase portrait of $E(X_2)$ in a neighborhood of $(1, \theta'_r)$. From the expression of the Jacobian matrices of $E(X_1)$ and $E(X_2)$ in these points (see (5)), we obtain that

$$\mathrm{sign}(f_1(\theta_i)) = \mathrm{sign}(f_2(\theta'_r)),$$
$$\mathrm{sign}(g'_1(\theta_i)) = \mathrm{sign}(g'_2(\theta'_r)),$$

where f_1 and g_1 (respectively f_2 and g_2) are the functions associated to X_1 (respectively X_2) according to (2). Therefore from (8) we deduce that

$$(\sigma_i, v_i) = (\sigma'_r, v'_r) \qquad \text{if} \quad h(1, \theta_i) = (1, \theta'_r). \tag{18}$$

Now we consider two cases depending on whether h preserves or reverses the orientation on ∂D. In the first case if $h(1, \theta_1) = (1, \theta'_p)$ then

$$h(1, \theta_i) = \begin{cases} (1, \theta'_{i+p-1}) & \text{if} \quad i+p-1 \leqslant 2k, \\ (1, \theta'_{i+p-1-2k}) & \text{if} \quad i+p-1 > 2k, \end{cases} \tag{19}$$

for $i=1, ..., 2k$. Since $(\sigma'_{i-2k}, v'_{i-2k})=(\sigma'_i, v'_i)$, from (18) and (19) it follows that

$$(\sigma_i, v_i)=(\sigma'_{i+p-1}, v'_{i+p-1}) \qquad \text{for} \quad i=1, ..., 2k.$$

As (σ, v) and (σ', v') are periodic sequences of period a divisor of $2k$, the above equation is true for all $i \in \mathbf{Z}$. Since $1 \leqslant p \leqslant 2k$, if we take $\tau=p-1$, then (16) holds.

On the other hand if h reverses the orientation and $h(1, \theta_1)=(1, \theta'_p)$, then

$$h(1, \theta_i)=\begin{cases}(1, \theta'_{p-i+1}) & \text{if} \quad p-i+1>0,\\ (1, \theta'_{p-i+1+2k}) & \text{if} \quad p-i+1 \leqslant 0,\end{cases} \tag{20}$$

for $i=1, ..., 2k$. As $(\sigma'_i, v'_i)=(\sigma'_{i+2k}, v'_{i+2k})$, according to (18) and (20) we obtain that

$$(\sigma_i, v_i)=(\sigma'_{2k-i+1+p}, v'_{2k-i+1+p}) \qquad \text{for} \quad i=1, ..., 2k.$$

From the periodicity of (σ, v) and (σ', v'), (17) follows with $\tau=p$, when $1 \leqslant p \leqslant 2k-1$. If $p=2k$ then (17) is also true with $\tau=0$ and the necessary condition of the proposition is proved.

In order to prove the sufficient condition, we note that according to Proposition 2, the phase portraits of $E(X_1)$ and $E(X_2)$ in $\bar{D}$ are an union of $2k$ elliptic, hyperbolic and parabolic sectors. We denote by $(1, \theta_i)$ and $(1, \theta'_i)$, $i=1, ..., 2k$ the critical points in ∂D of $E(X_1)$ and $E(X_2)$ respectively. Then we denote by R_i, $i=1, ..., 2k-1$, the sectors of the flow induced by $E(X_1)$ such that they are bounded by the straight lines $\theta=\theta_i$ and $\theta=\theta_{i+1}$, and denote by R_{2k} the sector which is bounded by $\theta=\theta_{2k}$ and $\theta=\theta_1$. The sectors R'_i, $i=1, ..., 2k$ of the flow induced by $E(X_2)$ are obtained in a similar way if we replace θ_i by θ'_i. Again by Proposition 2 the phase portrait of $E(X_1)$ (respectively $E(X_2)$) in the sector R_i (respectively R'_i) is completely determined by the values of (σ_i, v_i) and (σ_{i+1}, v_{i+1}) (respectively (σ'_i, v'_i) and $(\sigma'_{i+1}, v'_{i+1})$).

Assume that (16) holds and define

$$u(i)=\begin{cases}i+\tau & \text{if} \quad i+\tau \leqslant 2k,\\ i+\tau-2k & \text{if} \quad i+\tau>2k,\end{cases}$$

for $i=1, ..., 2k$, where τ verifies (16). If $i<2k$ then the critical points $(1, \theta'_{u(i)})$ and $(1, \theta'_{u(i+1)})$ are counterclockwise consecutive on ∂D and both together determine the sector $R'_{u(i)}$. Furthermore, according to (16), R_i and $R'_{u(i)}$ are sectors of the same topological type. We note that the above arguments are also valid when $i=2k$, if we replace $u(i+1)$ by $u(1)$.

Now we can define a homeomorphism (that preserves the orientation on ∂D) $h_i: R_i \to R'_{u(i)}$ such that it carries orbits of the flow induced by $E(X_1)$ onto orbits of the flow induced by $E(X_2)$, preserving their sense (for more details see for instance [Gb]). Next if we define $h: \bar{D} \to \bar{D}$ by

$$h(\rho, \theta) = h_i(\rho, \theta) \qquad \text{if} \quad (\rho, \theta) \in R_i,$$

then h provides a topological equivalence between $E(X_1)$ and $E(X_2)$ in $\bar{D}$ and so X_1 and X_2 are topologically equivalent.

On the other hand if there exists τ verifying (17) define

$$v(i) = \begin{cases} 2k - i + 1 + \tau & \text{if} \quad -i + \tau + 1 \leqslant 0, \\ -i + 1 + \tau & \text{if} \quad -i + \tau + 1 > 0, \end{cases}$$

for $i = 1, ..., 2k$. Now, if $i < 2k$ the critical points $(1, \theta'_{v(i)})$ and $(1, \theta'_{v(i+1)})$ are clockwise consecutive on ∂D. Therefore these points determine the sector $R'_{v(i+1)}$ of $E(X_2)$ and, according to (17), the orbits in R_i and $R'_{v(i+1)}$ are of the same type if we consider the clockwise orientation in $R'_{v(i+1)}$. So there exists a homeomorphism $h_i: R_i \to R'_{v(i+1)}$ (reversing the orientation) such that it carries orbits of the flow induced by $E(X_1)$ onto orbits of the flow induced by $E(X_2)$, preserving their sense. If $i = 2k$ then we replace $i+1$ by 1 to obtain an analogous homeomorphism $h_{2k}: R_{2k} \to R'_{v(1)}$. Finally if we define h from the h_i's as in the previous case, it follows again that X_1 and X_2 are topologically equivalent and the proposition is proved. ∎

We note that from Proposition 10, if $s = (\sigma, v) \in S^{2k}_m$, the sequence t defined by $t_i = s_{i+1}$ also belongs to S^{2k}_m. So we can define the application

$$\mathfrak{R}: S^{2k}_m \to S^{2k}_m$$

such that if $s = (\sigma, v) \in S^{2k}_m$, then $\mathfrak{R}(s)$ is the sequence of S^{2k}_m verifying

$$(\mathfrak{R}(s))_i = s_{i+1} \qquad \text{for all} \quad i \in \mathbf{Z}. \tag{21}$$

Whenever $f: A \to A$ is an arbitrary application then it is said that $a \in A$ is a *p-periodic point* of f when $f^p(a) = a$ and $f^i(a) \neq a$ for $i = 1, ..., p-1$ and p is called the *period* of a. The set $C \subset A$ is a *cycle of order p* (*p-cycle*) of f if there exists a p-periodic point $a \in A$ for f such that $C = \{a, f(a), ..., f^{p-1}(a)\}$. It is clear from (21) that $s \in S^{2k}_m$ is a p-periodic point of $\mathfrak{R}$ if and only if s is a p-periodic sequence. Therefore $C \subset S^{2k}_m$ is a p-cycle of $\mathfrak{R}$ if there exists a p-periodic sequence $s \in S^{2k}_m$ such that $C = \{s, \mathfrak{R}(s), ..., \mathfrak{R}^{p-1}(s)\}$. It is clear that each sequence $s \in S^{2k}_m$ belongs to some cycle of order p of $\mathfrak{R}$ where p is an even divisor of $2k$.

Next we define the application

$$\Psi\colon S_m^{2k} \to S_m^{2k},$$

where if $s \in S_m^{2k}$ then $\Psi(s)$ is given by

$$(\Psi(s))_i = s_{2k-i+1} \qquad \text{for all} \quad i \in \mathbf{Z}. \tag{22}$$

We note that Ψ is a well-defined application (see Proposition 10) that reverses the order of the elements $(\sigma_1, \nu_1), ..., (\sigma_{2k}, \nu_{2k})$ of s.

The following proposition shows some properties of the applications $\mathfrak{R}$ and Ψ.

PROPOSITION 12. *Let $m \in \mathbf{N}$, $0 \neq k \in J_m$ and assume that $\mathfrak{R}$ and Ψ are the applications on S_m^{2k} defined by* (21) *and* (22). *Then the following statements hold.*

(a) $\Psi \circ \Psi = Id$;

(b) $\Psi \circ \mathscr{R} = \mathscr{R}^{-1} \circ \Psi$.

Proof. From (22) we obtain that

$$[(\Psi \circ \Psi)(s)]_i = s_{2k-(2k-i+1)+1} = s_i \qquad \text{for all} \quad i \in \mathbf{Z},$$

and (a) follows. On the other hand according to (21) and (22) we have that

$$[(\Psi \circ \mathfrak{R})(s)]_i = [(\mathfrak{R}^{-1} \circ \Psi)(s)]_i = s_{2k-i+2} \qquad \text{for all} \quad i \in \mathbf{Z}.$$

Consequently we get (b). ∎

Now we give another interpretation of Proposition 11 through the applications $\mathfrak{R}$ and Ψ.

PROPOSITION 13. *Let $m \in \mathbf{N}$, $0 \neq k \in J_m$ and X_1, $X_2 \in \Omega_m^{2k}$. Denote by s and s' the sequences of S_m^{2k} associated to X_1 and X_2 respectively. Then X_1 and X_2 are topologically equivalent if and only if the sequences s and s' or s and $\Psi(s')$ belong to the same cycle of $\mathfrak{R}$.*

Proof. From Proposition 11, if we rewrite (16) and (17) by using the applications $\mathfrak{R}$ and Ψ, then we obtain that X_1 and X_2 are topologically equivalent if and only if there exists an integer τ such that $0 \leqslant \tau \leqslant 2k-1$ and $s = \mathfrak{R}^{\tau}(s')$ or $s = \mathfrak{R}^{-\tau}(\Psi(s'))$. Since the sequences s, s', and $\Psi(s')$ have the same period p a divisor of $2k$, the proposition follows. ∎

Now we define in S_m^{2k} the following equivalence relation: $s \sim \bar{s}$ if and only if one of the sequences s or $\Psi(s)$ belongs to the same cycle of $\mathfrak{R}$ that $\bar{s}$. According to Proposition 13, we have that the number of equivalence classes of $\sim$ in S_m^{2k} coincide with the number C_m^k of topological equivalence classes in Ω_m^{2k}.

In order to determine the equivalence class of a sequence $s \in S_m^{2k}$ with respect to $\sim$, we consider the cycle C of $\mathfrak{R}$ such that $s \in C$. From Proposition 12(b), $\Psi(\mathfrak{R}^i(s)) = \mathfrak{R}^{-i}(\Psi(s))$, and hence the cycle of $\Psi(s)$ is $C' = \{\Psi(t) : t \in C\}$. Therefore the equivalence class of s is $C \cup C'$.

We say that a cycle C of $\mathfrak{R}$ is *symmetrical* if $C = C'$, that is, if $\Psi(s) \in C$ for each $s \in C$. Then if we denote by D_m^k the number of cycles of $\mathfrak{R}$ in S_m^{2k} and denote by E_m^k the number of symmetrical cycles, we deduce that

$$C_m^k = E_m^k + \frac{D_m^k - E_m^k}{2} = \frac{E_m^k + D_m^k}{2}. \tag{23}$$

The following result determines the value of D_m^k.

Proposition 14. *Let $m \in \mathbf{N}$ and $0 \neq k \in J_m$.*

(a) *If m is odd and $k < m+1$, then*

$$D_m^k = \sum_{2t \mid k} P_{2t}, \qquad \textit{where} \quad P_{2t} = \frac{1}{t}\Big(2^{2t} - \sum_{\substack{r \mid t \\ r \neq t}} r P_{2r}\Big).$$

(b) *If m is even and $k < m+1$, then*

$$D_m^k = \sum_{t \mid k} P_{2t}, \qquad \textit{where} \quad P_{2t} = \frac{1}{t}\Big(2^{t} - \sum_{\substack{r \mid t \\ r \neq t}} r P_{2r}\Big).$$

(c) *If m is odd, then $D_m^{m+1} = (\sum_{2t \mid m+1} P_{2t}) - 1$, with P_{2t} defined as in* (a).

(d) *If m is even, then $D_m^{m+1} = (\sum_{t \mid m+1} P_{2t}) - 1)$, with P_{2t} defined as in* (b).

Proof. To find D_m^k notice that each q-periodic sequence $s \in S_m^{2k}$ belongs to a q-cycle of $\mathfrak{R}$. Thus if P_q denotes the number of q-cycles of $\mathfrak{R}$ in S_m^{2k} then $P_q = N_q / q$, where N_q is the number of q-periodic sequences in S_m^{2k}.

Let m be odd. By Proposition 4, k is even and, from Proposition 7(b), the periods of the sequences in S_m^{2k} are even divisors of k. Let $q = 2t$ be one of these periods. Each $2t$-periodic sequence is characterized by the elements $(\sigma_1, v_1), ..., (\sigma_{2t}, v_{2t})$ and if $k < m+1$, by Proposition 10(a), there exist 2^{2t+1} ways of choosing these elements. Nevertheless it is necessary to remove those sequences with period an even proper divisor of $2t$. Hence,

$$N_{2t} = 2^{2t+1} - \sum_{\substack{r \mid t \\ r \neq t}} N_{2r} \qquad \text{and} \qquad P_{2t} = \frac{1}{t}\Big(2^{2t} - \sum_{\substack{r \mid t \\ r \neq t}} r P_{2r}\Big).$$

By adding P_{2t} for all divisors $2t$ of k, (a) follows.

Let m be even. In this case, from Proposition 4, k is odd and the periods of the sequences of S_m^{2k} are the even divisors of $2k$. If $k < m+1$ and $2t$ is one of these divisors, according to Proposition 7(b) and 10(a), there exist just 2^{t+1} ways of choosing the elements $(\sigma_1, v_1), ..., (\sigma_{2t}, v_{2t})$. Therefore

$$N_{2t} = 2^{t+1} - \sum_{\substack{r \mid t \\ r \neq t}} N_{2r} \qquad \text{and} \qquad P_{2t} = \frac{1}{t}\Big(2^t - \sum_{\substack{r \mid t \\ r \neq t}} rP_{2r}\Big).$$

Now (b) holds by adding P_{2t} for all divisors $2t$ of $2k$.

Lastly, we remark that the above computations are valid for the case $k = m+1$, but, by Proposition 10(b), we have to rule out the two sequences such that $\sigma_j = v_j$ for all $j \in \mathbf{Z}$. These sequences belong to the same 2-cycle of $\mathfrak{R}$ and hence statement (c) and (d) follow. ∎

Next we obtain a characterization of the symmetrical cycles which will be useful in determining E_m^k.

PROPOSITION 15. *Set $m \in \mathbf{N}$ and $0 \neq k \in J_m$. Let $\mathfrak{R}$ be the application defined on S_m^{2k} by* (21) *and let C be a cycle of $\mathfrak{R}$. Then C is symmetrical if and only if there exists $s \in C$ such that $\Psi(s) = \mathfrak{R}(s)$.*

Proof. To confirm the sufficient condition note that if there exists $s \in C$ such that $\Psi(s) = \mathfrak{R}(s)$ then $\Psi(s) \in C$ and, according to Proposition 12(b), $\Psi(t) \in C$ for every $t \in C$; this is, the cycle is symmetrical.

Now suppose that C is a symmetrical p-cycle of $\mathfrak{R}$ and let $s \in C$. Consequently, there exists j such that $0 \leqslant j \leqslant p-1$ and $\Psi(s) = \mathfrak{R}^j(s)$, or equivalently from (21) and (22), we get that

$$s_{2k-i+1} = s_{i+j} \qquad \text{for all} \quad i \in \mathbf{Z}. \tag{24}$$

Remember that $s = (\sigma, v)$ satisfies $\sigma_i \sigma_{i+1} < 0$, for all $i \in \mathbf{Z}$, and hence $s_1 \neq s_l$ when l is even. Now we take $i = 2k$ in (24). Then it follows that $s_1 = s_j$ and so j is necessarily odd. Therefore we choose $\bar{s} = \mathfrak{R}^{(j-1)/2}(s) \in C$ and, by taking into account Proposition 12(b), we obtain

$$\Psi(\bar{s}) = \Psi(\mathfrak{R}^{(j-1)/2}(s)) = \mathfrak{R}^{(1-j)/2}(\Psi(s)) = \mathfrak{R}^{(j+1)/2}(s) = \mathfrak{R}(\bar{s}),$$

and the proof is finished. ∎

COROLLARY 16. *Under the assumptions of Proposition* 15 *if C is a symmetrical cycle, then there exist exactly two elements of C satisfying $\Psi(s) = \mathfrak{R}(s)$.*

Proof. Let C be a symmetrical cycle of $\mathfrak{R}$ of period p. From Proposition 15 there exists at least one element $\bar{s} \in C$ such that $\Psi(\bar{s}) = \mathfrak{R}(\bar{s})$. Since every element $s \in C$ is given by $\mathfrak{R}^i(\bar{s})$ for $0 \leqslant i \leqslant p-1$, then there exists another $s \in C$ satisfying $\Psi(s) = \mathfrak{R}(s)$ if and only if there exists $i \in \{1, ..., p-1\}$ such that $\Psi(\mathfrak{R}^i(\bar{s})) = \mathfrak{R}^{i+1}(\bar{s})$. By Proposition 12(b) this equation is equivalent to $\mathfrak{R}^{-i}\Psi(\bar{s}) = \mathfrak{R}^{i+1}(\bar{s})$. Finally $\Psi(\bar{s}) = \mathfrak{R}(\bar{s})$ implies $\mathfrak{R}^{2i+1}(\bar{s}) = \mathfrak{R}(\bar{s})$, and this equality admits the single solution $i = p/2$. So $\bar{s}$ and $\mathfrak{R}^{p/2}(\bar{s})$ are the only elements s of C that verify $\Psi(s) = \mathfrak{R}(s)$. ∎

Now we can calculate the value of E_m^k.

PROPOSITION 17. *Let $m \in \mathbf{N}$ and $0 \neq k \in J_m$.*

(a) *If m is odd and $k < m+1$, then*

$$E_m^k = \sum_{2t \mid k} I_{2t}, \qquad \text{where} \quad I_{2t} = 2^{t+1} - \sum_{\substack{r \mid t \\ r \neq t}} I_{2r}.$$

(b) *If m is even and $k < m+1$, then*

$$E_m^k = \sum_{t \mid k} I_{2t}, \qquad \text{where} \quad I_{2t} = 2^{(t+1)/2} - \sum_{\substack{r \mid t \\ r \neq t}} I_{2r}.$$

(c) *If m is odd, then $E_m^{m+1} = (\sum_{2t \mid m+1} I_{2t}) - 1$, with I_{2t} defined as in* (a).

(d) *If m is even, $E_m^{m+1} = (\sum_{t \mid m+1} I_{2t}) - 1$, with I_{2t} defined as in* (b).

Proof. From Proposition 15 and Corollary 16, the number I_q of symmetrical q-cycles of $\mathfrak{R}$ will be obtained if we divide by 2 the number of the q-periodic sequences s in S_m^{2k} such that $\Psi(s) = \mathfrak{R}(s)$. These sequences verify (see (21) and (22)) $s_{2k-i+1} = s_{i+1}$ for all $i \in \mathbf{Z}$ and since q is a divisor of $2k$ we obtain that

$$s_{q-i+1} = s_{i+1} \qquad \text{for all} \quad i \in \mathbf{Z}. \tag{25}$$

Therefore $s_{(q/2)+1+i} = s_{(q/2)-i+1}$ for $i = 1, ..., (q/2)-1$, and so the values $s_1, ..., s_{(q/2)+1}$ determine completely the sequence s.

Now let m be odd. In this case we know that the periods of the sequences in S_m^{2k} are $q = 2t$, where $2t$ is a divisor of k (see Proposition 7(b)). Assuming that $k < m+1$, by Proposition 10(a) there exist 2^{t+2} ways of choosing the elements $s_1, ..., s_{t+1}$ of each sequence verifying (25) for $q = 2t$. If we remove the sequences with period a proper divisor $2r$ of $2t$ and divide the result by 2, then it follows that:

$$I_{2t} = 2^{t+1} - \sum_{\substack{r \mid t \\ r \neq t}} I_{2r}.$$

By adding I_{2t} for all the divisors $2t$ of k, we conclude (a).

Next we assume that m is even, then the periods of the sequences in S_m^{2k} are the even divisors $2t$ of $2k$ (see Proposition 7(b)). If $k<m+1$ from Proposition 7 we know that $s_{i+k}=-s_i$ for all $i\in\mathbf{Z}$. Note that if s is $2t$-periodic then this relation implies that

$$s_{i+t}=-s_i, \qquad \text{for all} \quad i\in\mathbf{Z}. \tag{26}$$

From (25) (by taking $q=2t$) and (26) it follows that $s_{i+1}=s_{2t-i+1}=-s_{t-i+1}$ for all $i\in\mathbf{Z}$. So $s_{(t+1)/2+i}=-s_{(t+1)/2-i+1}$ for $i=1, ..., (t-1)/2$. Then, in order to determine elements $s_1, ..., s_t$ it is enough to know elements $s_1, ..., s_{(t+1)/2}$. If $k<m+1$ with the help of Proposition 10(a) we can take $2^{1+(t+1)/2}$ different sequences in S_m^{2k} verifying (25). If we ignore the sequences with period a proper divisor $2r$ of $2t$ and divide the result by 2, then we obtain that

$$I_{2t}=2^{(t+1)/2}-\sum_{\substack{r\mid t\\ r\neq t}} I_{2r}.$$

Finally statement (b) is proved if we add I_{2t} for all the divisors of $2k$.

Lastly since the sequences such that $\sigma_j=v_j$ for all $j\in\mathbf{Z}$ verify $\Psi(s)=\mathfrak{R}(s)$, the proof of statements (c) and (d) are analogous to the corresponding ones of Proposition 14. ∎

Note that the values of D_m^k and E_m^k determine the value of C_m^k (see (23)) and therefore we can prove Theorem B. This proof is the main goal of this section.

Proof of Theorem B. From Corollary 5 it follows that

$$C_m=\sum_{k\in J_m} C_m^k, \tag{27}$$

where $J_m=\{k=2l : 0\leqslant l\leqslant(m+1)/2\}$ if m is odd and $J_m=\{k=2l+1 : 0\leqslant l\leqslant m/2\}$ if m is even.

For $k\neq 0$ we obtain C_m^k from D_m^k and E_m^k from (23). Since $C_m^0=2$, from Propositions 14 and 17 and (27) the theorem follows. ∎

5. Local Phase Portrait at a Critical Point

We consider the system of differential equations given by

$$\frac{dx}{dt}=\sum_{i\geqslant m} P_i(x, y), \qquad \frac{dy}{dt}=\sum_{i\geqslant m} Q_i(x, y), \tag{28}$$

where P_i and Q_i are homogeneous polynomials of degree i in the variables x and y. The expression of system (28) in polar coordinates is

$$\frac{dr}{dt} = \sum_{i \geqslant m} r^i f_i(\theta), \qquad \frac{d\theta}{dt} = \sum_{i \geqslant m} r^{i-1} g_i(\theta),$$

where

$$\begin{aligned} f_i(\theta) &= \cos\theta P_i(\cos\theta, \operatorname{sen}\theta) + \operatorname{sen}\theta Q_i(\cos\theta, \operatorname{sen}\theta), \\ g_i(\theta) &= \cos\theta Q_i(\cos\theta, \operatorname{sen}\theta) - \operatorname{sen}\theta P_i(\cos\theta, \operatorname{sen}\theta). \end{aligned} \tag{29}$$

If we introduce a new time s via $ds/dt = r^{m-1}$ then the above system becomes

$$r' = \sum_{i \geqslant m} r^{i-m+1} f_i(\theta), \qquad \theta' = \sum_{i \geqslant m} r^{i-m} g_i(\theta), \tag{30}$$

where the prime denotes derivative with respect s.

If we do the same transformations in the system

$$\frac{dx}{dt} = P_m(x, y), \qquad \frac{dy}{dt} = Q_m(x, y),$$

we obtain system

$$r' = r f_m(\theta), \qquad \theta' = g_m(\theta), \tag{31}$$

where f_m and g_m are defined in (29).

Proof of Theorem C. First we consider that X_m is structurally stable with respect to perturbations in H_m and distinguish the two cases of Theorem A.

Case 1. $E(X_m)$ has no critical points on ∂D and $I_{X_m} \neq 0$. By Proposition 3(a) $E(X_m)$ has no critical points on $\rho = 0$ and this circle is a stable or unstable limit cycle. Since the system associated to $E(X_m)$ (see (3)) was obtained from (31) via the change $\rho = r/(1+r)$, the circle $r = 0$ is also a stable or unstable limit cycle for system (31).

The critical points of (30) and (31) on $r = 0$ are determined by the zeros of $g_m(\theta)$, and so $r = 0$ is also a periodic orbit for (30). Furthermore, since the dominant terms of (30) in a neighborhood of $r = 0$ are precisely $r f_m(\theta)$ and $g_m(\theta)$, the orbit $r = 0$ is a limit cycle with the same type of stability for (30) and (31). Therefore the origin of $\mathbf{R}^2$ is a focus with the same type of stability for X and X_m and the theorem is proved in this case.

Case 2. $E(X_m)$ has critical points on ∂D and all these points are hyperbolic. From Proposition 3(b), system (31) also has critical points on $r=0$ and they are hyperbolic.

Comparing (30) and (31) we observe that the critical points on $r=0$ are the same for both systems. Furthermore the Jacobian matrices of these systems in one critical point $(0, \theta^*)$ are

$$\begin{pmatrix} f_m(\theta^*) & 0 \\ g_{m+1}(\theta^*) & g'_m(\theta^*) \end{pmatrix}$$

and

$$\begin{pmatrix} f_m(\theta^*) & 0 \\ 0 & g'_m(\theta^*) \end{pmatrix}$$

respectively. So all the critical points of (30) on $r=0$ are hyperbolic and the local phase portraits of (30) and (31) in $(0, \theta^*)$ are topologically equivalent. Since all these points are isolated critical points, they determine the phase portraits of both systems in a neighborhood of $r=0$. Hence X and X_m are locally topologically equivalent at the origin. ∎

In order to state that the converse of Theorem C is not true, we need to use some techniques which appear in the work of Brunella and Miari [BM]. We reproduce here these results.

Let X be an analytical vector field in $\mathbf{R}^2$ such that $X(0)=0$. The system associated to X can be expressed

$$x' = P(x, y) = \sum_{i+j \in \mathbf{N}} a_{ij} x^i y^j, \qquad y' = Q(x, y) = \sum_{i+j \in \mathbf{N}} b_{ij} x^i y^j.$$

We introduce the following subset of $\mathbf{Z}^2$:

$$T = \{(i, j+1) : a_{ij} \neq 0\} \cup \{(i+1, j) : b_{ij} \neq 0\}. \tag{32}$$

Then the *Newton polygon* of the vector field X is the convex envelope of the set

$$\bigcup_{(i,j) \in T} \{(i+x, j+y) : \forall x, y \in [0, +\infty)\}.$$

The *Newton diagram* Γ of X is the union of all the compact edges γ_k of the Newton polygon.

The polynomial vector field $X_\Delta = (P_\Delta, Q_\Delta)$ where

$$P_\Delta = \sum_{(i, j+1) \in \Gamma} a_{ij} x^i y^j, \qquad Q_\Delta = \sum_{(i+1, j) \in \Gamma} b_{ij} x^i y^j,$$

is called the *principal part* of vector field X and the polynomial vector field $X_{\gamma_k} = (P_{\gamma_k}, Q_{\gamma_k})$ where

$$P_{\gamma_k} = \sum_{(i,\, j+1) \in \gamma_k} a_{ij} x^i y^j, \qquad Q_{\gamma_k} = \sum_{(i+1,\, j) \in \gamma_k} b_{ij} x^i y^j,$$

is called the *quasi-homogeneous component* of the principal part X_Δ relative to γ_k.

We shall say that two analytical vector fields X, Y on $\mathbf{R}^2$ satisfying $X(0) = Y(0) = 0$ are *locally topologically equivalent modulo center-focus at the origin* if either X and Y are locally topologically equivalent at 0, or 0 is a center or a focus for X and Y.

THEOREM 18. *Let X be an analytical vector field on $\mathbf{R}^2$ such that $X(0) = 0$. If there exists a quasi-homogeneous component X_γ of the principal part X_Δ such that 0 is an isolated singularity of X_γ, then X and X_γ are locally topologically equivalent modulo center-focus at 0.*

Proof. See Theorem B of [BM]. ∎

Now we apply this result to prove that the converse of Theorem C is not true.

PROPOSITION 19. *For each $m \in \mathbf{N}$ there exists some vector field $X_m \in H_m$ which is not structurally stable with respect to perturbations in H_m such that the phase portraits of $X_m = (P_m, Q_m)$ and $X = (\sum_{i \geqslant m} P_i, \sum_{i \geqslant m} Q_i)$ are locally topologically equivalent at the origin.*

Proof. Fix $m \in \mathbf{N}$ and consider the vector field $X_m = (P_m, Q_m)$ associated to the system given by

$$x' = -y^m, \qquad y' = \frac{[(x+y)^{m+1} - y^{m+1}]}{x}. \tag{33}$$

From (7) the function $\bar{g}$ associated to X_m is $\bar{g}(\lambda) = (1+\lambda)^{m+1}$. Its only zero is -1 with multiplicity $m+1$, and so the corresponding critical points of $E(X_m)$ on ∂D are $(1, -\pi/4)$ and $(1, 3\pi/4)$ that verify that $g'(\theta) = 0$ when $\theta = -\pi/4$, $3\pi/4$. Hence, by Proposition 6, X_m is not structurally stable with respect to perturbations in H_m.

Now we consider the vector field $X = (\sum_{i \geqslant m} P_i, \sum_{i \geqslant m} Q_i)$, where P_i and Q_i are homogeneous polynomials of degree i and P_m and Q_m are defined in (33). The set T associated to X (see (32)) contains the points $(0, m+1)$ and $(m+1, 0)$, and therefore the only compact edge of Newton polygon of X is

$$\gamma = \{(x, y) \in \mathbf{R}^2 : x, y \geqslant 0, x + y = m+1\}.$$

So the only quasi-homogeneous component of the principal part of X is $X_\gamma = X_m$. Finally, since the origin of X_m is an isolated critical point and is not a center or a focus (the straight line of slope -1 is invariant under the flow of X_m), we can apply Theorem 18 to obtain that X and X_m are locally topologically equivalent at the origin. ∎

6. Local Phase Portrait at Infinity

We consider the system of differential equations given by

$$\frac{dx}{dt} = \sum_{i=0}^{m} P_i(x, y), \qquad \frac{dy}{dt} = \sum_{i=0}^{m} Q_i(x, y), \tag{34}$$

where P_i and Q_i are homogeneous polynomials of degree i in the variables x and y. System (34) is expressed in polar coordinates by the equations

$$\frac{dr}{dt} = \sum_{i=0}^{m} r^i f_i(\theta), \qquad \frac{d\theta}{dt} = \sum_{i=0}^{m} r^{i-1} g_i(\theta),$$

where f_i and g_i are defined as in (29).

The transformation $\delta = 1/r$ writes the previous system as

$$\frac{d\delta}{dt} = -\sum_{i=0}^{m} \frac{1}{\delta^{i-2}} f_i(\theta), \qquad \frac{d\theta}{dt} = \sum_{i=0}^{m} \frac{1}{\delta^{i-1}} g_i(\theta).$$

Finally if we consider a new time s such that $dt/ds = \delta^{m-1}$, the above system becomes

$$\begin{aligned} \delta' &= -[\delta f_m(\theta) + \delta^2 f_{m-1}(\theta) + \cdots + \delta^{m+1} f_0(\theta)], \\ \theta' &= g_m(\theta) + \delta g_{m-1}(\theta) + \cdots + \delta^m g_0(\theta). \end{aligned} \tag{35}$$

After these changes of variables the proof of Theorem E is analogous to the proof of Theorem C.

Proof of Theorem E. The infinity of X corresponds to the invariant circle $\delta = 0$ of (35). Now if we apply to X_m the changes of variable that transform (34) into (35), then we obtain

$$\delta' = -\delta f_m(\theta), \qquad \theta' = g_m(\theta). \tag{36}$$

Since X_m is structurally stable, the invariant circle $\delta = 0$ of (36) (corresponding to ∂D for $E(X_m)$) is either an attractor or repulsor limit cycle, or it contains a finite number of hyperbolic critical points. Now we observe that system (35) is obtained from (36) by adding terms of higher degree in δ. Therefore, by using the same arguments as in the proof of Theorem C, the induced flows by (35) and (36) are topologically equivalent in a neighborhood of $\delta = 0$ and the theorem follows. ∎

Due to the analogy between the origin and the infinity, note finally that the converse of Theorem E is false in the same way as it was the converse of Theorem C.

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