COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Limit cycles of polynomial Liénard systems

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Recently [H. Giacomini and S. Neukirch, Phys. Rev. E **56**, 3809 (1997)], an algorithm to obtain the number of limit cycles of Liénard systems has been proposed. The quoted paper also includes a method to approximate the eventual limit cycles and a conjecture on the behavior of the algorithm. The algorithm is reviewed and some examples, which show that the algorithm is really efficient, are given. However, these examples indicate that the aforementioned conjecture may have been incorrectly stated. A different conjecture is proposed and some open questions are formulated. [S1063-651X(98)16809-5]

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I. INTRODUCTION

In this Comment, we are concerned with the family of Liénard systems

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{1.1}$$

where $F(x)$ is an odd polynomial. As usual, the dot denotes a derivative with respect to the time t. Obviously, these systems have only one equilibrium at the origin. As it is well known, systems (1.1) are a particular case of the more general Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

for it suffices to take $F(x) = \int_0^x f(s)\, ds$, $g(x) = x$, and $y = \dot{x} + F(x)$.

Recently, Giacomini and Neukirch [2,3] have developed an algorithm to determine the number of limit cycles of system (1.1), along with a method to approximate such limit cycles by means of algebraic curves. It is remarkable that the Giacomini-Neukirch algorithm is nonperturbative and seems to work very well. However, it lacks a firm theoretical basis and so it still needs additional research in order to clarify its possibilities and general scope.

In this paper, we first review the Giacomini-Neukirch algorithm. After that, we give some examples that seem to indicate that a conjecture related with the algorithm should be corrected as indicated below, and formulate some open questions about the algorithm.

II. THE GIACOMINI-NEUKIRCH ALGORITHM

As the quoted algorithm is mainly explained by examples (see [2,3]), in order to be more precise the following result will be useful. We remark that our notation differs slightly from that used in [3]. In what follows, the prime will denote a derivative with respect to the variable x.

Proposition 1. Consider a Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{2.1}$$

and, for $k \in \mathbb{N}$, define functions $\varphi_0, \varphi_1, \ldots, \varphi_{2k}$ with the following properties:

$$\varphi_0(x) = 1,$$
$$\varphi_1'(x) = 0, \quad \varphi_2'(x) = 2kg(x), \quad \varphi_3'(x) = F(x)\varphi_2'(x),$$
$$\varphi_j'(x) = F(x)\varphi_{j-1}'(x) + (2k-j+2)g(x)\varphi_{j-2}(x), \tag{2.2}$$
$$j = 4,5,\ldots,2k.$$

Then the function

$$V_k(x,y) = \sum_{j=0}^{2k} \varphi_j(x) y^{2k-j}$$

verifies

$$\dot{V}_k(x,y) = -R_k(x),$$

where

