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HORSESHOES NEAR HOMOCLINIC ORBITS FOR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$

JAUME LLIBRE
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08913 Bellaterra, Barcelona, Spain
jllibre@mat.uab.cat

ENRIQUE PONCE
Departamento de Matemática Aplicada II, E. S. Ingenieros, Universidad de Sevilla, 41092 Sevilla, Spain
eponcem@us.es

ANTONIO E. TERUEL
Departament de Matemàtiques i Informàtica, Universitat de Illes Balears, 07122 Palma, Spain
antonioe.teruel@uib.es

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For a three-parametric family of continuous piecewise linear differential systems introduced by Arneodo *et al.* [1981] and considering a situation which is reminiscent of the Hopf-Zero bifurcation, an analytical proof on the existence of a two-parametric family of homoclinic orbits is provided. These homoclinic orbits exist both under Shil'nikov ($0 < \delta < 1$) and non-Shil'nikov assumptions ($\delta \geq 1$). As it is well known for the case of differentiable systems, under Shil'nikov assumptions there exist infinitely many periodic orbits accumulating to the homoclinic loop. We also prove that this behavior persists at $\delta = 1$. Moreover, for $\delta > 1$ and sufficiently close to 1 we show that these periodic orbits persist but then they do not accumulate to the homoclinic orbit.

Keywords: Homoclinic orbits; piecewise linear differential systems; horseshoes.

1. Introduction

It is well known that three-dimensional differential systems can exhibit chaotic dynamics. In some specific cases, homoclinic loops (invariant closed curves with exactly one singular point) act as organizing centers of such complex dynamical behavior. In fact, the celebrated paper of Shil'nikov [Shil'nikov, 1965] guarantees the existence of infinitely many unstable periodic orbits in every neighborhood of a homoclinic orbit associated to a saddle-focus equilibrium point under certain hypotheses on the eigenvalues of its linearization. More precisely, if λ and $-\lambda\delta \pm i\omega$ are the eigenvalues of the saddle-focus point, the Shil'nikov case requires that $0 < \delta < 1$. The ratio δ is strongly related to the saddle quantity σ quoted in Shil'nikov's and Belyakov's works, see [Kuznetsov, 2004; Shil'nikov *et al.*, 2001], and references therein.

Later on, in [Shil'nikov, 1970] the same author shows that under the same hypotheses the dynamics associated to the existence of the homoclinic orbit is that of a Birkhoff–Morse system (conjugated to a shift with infinitely many symbols). The richness of the structure of periodic orbits around a homoclinic orbit of Shil'nikov type was analyzed by Belyakov [1974, 1980, 1984], Glendinning and Sparrow [1984] and Gaspard *et al.* [1984]. In any case, the application of Shil'nikov theorems needs firstly to show