Physica D 409 (2020) 132523

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Integrability by separation of variables

Jaume Llibre [a,*], Rafael Ramírez [b]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[b] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda dels Països Catalans 26, 43007 Tarragona, Catalonia, Spain

ARTICLE INFO

Article history:
Received 29 September 2018
Received in revised form 10 April 2020
Accepted 11 April 2020
Available online 27 April 2020
Communicated by T. Sauer

Keywords:
Separation of variables
Hamiltonian system
Levi-Civita conditions
Stark problem
Integrability in the Jacobi sense
Integrability in the Liouville sense

ABSTRACT

We study the integrability in the Jacobi sense (integrability by separation of variables), of the Hamiltonian differential systems using the Levi-Civita Theorem. In particular we solve the Stark problem for $N > 3$.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

The study of the integrability by separation of variables of the Hamilton–Jacobi equations is a classical problem in Mechanics, dating back to the foundational works of Jacobi, Stäckel, Levi-Civita and others.

In 1904 and in a letter addressed to P. Stäckel and published in the Matematische Annalen [1], Levi-Civita deals with the problem of the integration by separation of variables. In the introduction of this letter he writtes: *Ho notato che si possono facilmente assegnare (sotto forma esplicita di equazioni a derivate parziali ...) le condizioni necessarie e sufficienti cui deve soddisfare una H affinch'e l'equazione*

$$H\left(z_1,\ldots,z_M,\frac{\partial W}{\partial z_1},\ldots,\frac{\partial W}{\partial z_M}\right)=h, \tag{1}$$

ammetta un integrale completo della forma

$$W=W_1(z_1,\alpha_1,\ldots,\alpha_M)+\cdots+W_M(z_M,\alpha_1,\ldots,\alpha_M), \tag{2}$$

dove $\alpha_1,\ldots,\alpha_M$ and h sono le costanti arbitrarie. Da queste condizioni scaturiscono alcune conseguenze di indole generale, che mi sembrano abbastanza interessanti, per quanto il dedurre da esse la completa risoluzione del problema apparisca ancora laborioso, e non vi sia nemmeno - oserei affermare - grande speranza di trovare tipi essenzialmente nuovi, oltre a quelli da Lei Stắckel scoperti. Indeed, Levi-Civita shows that

Theorem 1 (*Levi-Civita Theorem*). *Hamilton–Jacobi equation (1) has a first integral of the form (2), if and only if the Hamiltonian H satisfy the $M(M-1)/2$ second-order partial differential equations*

$$\begin{aligned}L_{jk}(z,P):=\;&\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_j\partial z_k}+\frac{\partial H}{\partial z_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_j\partial P_k}\\&-\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial z_j\partial P_k}-\frac{\partial H}{\partial P_k}\frac{\partial H}{\partial z_j}\frac{\partial^2 H}{\partial P_j\partial z_k}=0,\end{aligned} \tag{3}$$

for $j,k=1,\ldots,M$ with $j\neq k$. Take into account that $L_{jk}(z,P)=L_{kj}(z,P)$.

* Corresponding author.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), rafaelorlando.ramirez@urv.cat (R. Ramírez).

https://doi.org/10.1016/j.physd.2020.132523
0167-2789/© 2020 Elsevier B.V. All rights reserved.